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Interior Regularity Estimates in High Conductivity Homogenization and Application

Abstract In this paper, uniform pointwise regularity estimates for the solutions of conductivity equations are obtained in a unit conductivity medium reinforced by a ε-periodic lattice of highly conducting thin rods. The estimates are derived only at a distance $\varepsilon^{1+\tau}$ (for some $\tau > 0$) away from the fibres. This distance constraint is rather sharp since the gradients of the solutions are shown to be unbounded locally in L^p as soon as $p > 2$. One key ingredient is the derivation in dimension two of regularity estimates to the solutions of the equations deduced from a Fourier series expansion with respect to the fibres direction, and weighted by the high-contrast conductivity. The dependence on powers of ε of these two-dimensional estimates is shown to be sharp. The initial motivation for this work comes from imaging, and enhanced resolution phenomena observed experimentally in the presence of micro-structures [23]. We use these regularity estimates to characterize the signature of low volume fraction heterogeneities in the fibred reinforced medium assuming that the heterogeneities stay at a distance $\varepsilon^{1+\tau}$ away from the fibres.

1 Introduction

Consider a material contained in $\Omega = \omega \times (-L, L) \subset \mathbb{R}^3$ where ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\omega$. Given some fixed $\omega_0 \Subset \omega$, we assume that inside $\Omega_0 = \omega_0 \times (-L, L)$, the material contains small cylindrical rods of high conductivity. A sketch of the domain is represented

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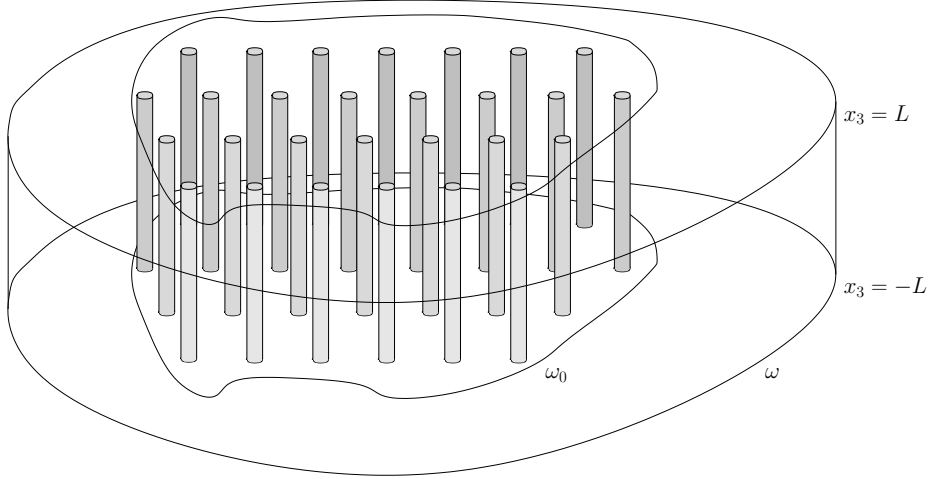
Fig. 1 Sketch of a domain $\Omega = \omega \times (-L, L)$ containing cylindrical rods.

in Figure 1. For $\varepsilon > 0$, $r_\varepsilon \ll \varepsilon$, and $(m, n) \in \mathbb{Z}^2$, let

$$D_{m,n,\varepsilon} = D_{\varepsilon r_\varepsilon}(m\varepsilon, n\varepsilon) \,,$$
$$Q_{m,n,\varepsilon} = D_{m,n,\varepsilon} \times (-L, L) \,,$$

where $D_{\varepsilon r_\varepsilon}(m\varepsilon, n\varepsilon)$ is the disc of radius $\varepsilon r_\varepsilon$ centered at $(m\varepsilon, n\varepsilon)$. Introduce the index set

$$I_\varepsilon = \{(m, n) \in \mathbb{Z}^2 : D_{m,n,\varepsilon} \subset \omega_0\} \,,$$

and set

$$D_\varepsilon = \bigcup_{(m,n) \in I_\varepsilon} D_{m,n,\varepsilon} \,, \qquad Q_\varepsilon = \bigcup_{(m,n) \in I_\varepsilon} Q_{m,n,\varepsilon} \,.$$

We will assume that the conductivity parameter of Ω is of the form

$$a_\varepsilon = \begin{cases} 1 & \text{in } \Omega \setminus Q_\varepsilon \,, \\ \alpha_\varepsilon & \text{in } Q_\varepsilon \,, \end{cases} \tag{1}$$

where

$$0 < \kappa_- \leq \alpha_\varepsilon \, \pi r_\varepsilon^2 \leq \kappa_+ \,. \tag{2}$$

Additionally, we assume

$$0 < \frac{2\pi}{\gamma_+} \leq -\varepsilon^2 \ln r_\varepsilon \leq \frac{2\pi}{\gamma_-} \,. \tag{3}$$

We consider, for $\Phi_b \in C^1(\bar{\Omega})$ and $F \in L^2(\Omega)$, the solution $\Psi_\varepsilon \in H^1(\Omega)$ of

$$\begin{cases} -\text{div}(a_\varepsilon \, \nabla \Psi_\varepsilon) = F & \text{in } \Omega \,, \\ \Psi_\varepsilon = \Phi_b & \text{on } \partial\Omega \,. \end{cases} \tag{4}$$

This model, initially introduced by Fenchenko & Khruslov [19], has been studied in the context of homogenization by several authors [21,5,12,9,27]. It is known to have a non-standard behaviour when ε tends to zero. Namely, the homogenized limit is not of divergence form, and admits a non-local term (see Theorem 3 for the precise form of the limit). While it is clear that, thanks to the

ellipticity, a global $W^{1,2}(\Omega)$ bound holds, one can show that except for special boundary data Φ_b, the solutions Ψ_ε of (4) are unbounded in $W^{1,p}_{\mathrm{loc}}(\Omega_0)$ for any $p > 2$, see Corollary 2. This makes the situation very different from the case of bounded coefficients: Meyers' Theorem [28] shows that solutions are bounded in $W^{1,p}_{\mathrm{loc}}$ for some $p > 2$ in that case. In the case of periodic composites with bounded coefficients, Li & Vogelius [25] and Li & Nirenberg [24] showed one can in fact obtain $W^{1,\infty}$ estimates. However, such improved regularity estimates strongly depend on the contrast of the coefficients as shown for example in [22, 3, 4].

The first goal of this work is to establish interior $C^{1,\alpha}$ estimates, uniformly in ε, for Ψ_ε away from Q_ε. We show that in a set $\Omega_\varepsilon = \omega_\varepsilon \times (-l, l)$ improved regularity estimates can be obtained. The set ω_ε is "almost" the complement of the high conductivity fibres in the sense that $\omega_\varepsilon \cap D_\varepsilon = \emptyset$ and $\omega_0 \setminus \omega_\varepsilon$ tends to zero with ε. Introducing V the solution in $H^1_0(\Omega)$ of

$$-\Delta V = F,$$

we show that

$$\|\Psi_\varepsilon - V\|_{C^{1,\nu}(\Omega_\varepsilon)} \le C \|F\|_{L^2(\Omega)},$$

see Theorems 1 and 2 for precise statements.

We can think of several applications for this work. One could for example use this result to establish a posteriori error estimates for the numerical solutions of (4). Our initial motivation came from a question related to imaging. In a recent work, Ammari et al. [1] showed that the signature of small inclusions inside a periodic medium was determined by the effective properties of this medium, in the limit when both the period and the size of the inclusions tend to zero. The motivation for this study was to provide a mathematical perspective on the so-called "resolution beyond the diffraction limit" verified experimentally [23].

Recent developments in material sciences, see Bouchitté et al. [7], have shown that very high contrast composite materials, with scalings similar to the ones used in this paper, could be used to construct meta-materials with particularly interesting properties (such as materials with negative optical indices). Such composites are out of the scope of [1]. This work relies on elliptic regularity estimates shown by Li & Nirenberg which do not apply for large contrast. These imaging problems involve either the Helmholtz equation, or the full Maxwell equation. We limited our study to the case of a real valued conductivity coefficient in this work. Assume a perturbation of small volume G_ε is located inside Ω_ε with conductivity $\gamma_1 \in C^0(\Omega)$. The conductivity of the defective medium is

$$a_{\varepsilon,d} = \begin{cases} 1 & \text{in } \Omega \setminus (Q_\varepsilon \cup G_\varepsilon) , \\ \gamma_1 & \text{in } G_\varepsilon , \\ \alpha_\varepsilon & \text{in } Q_\varepsilon . \end{cases} \tag{5}$$

For $\Phi_b \in C^1(\bar{\Omega})$, consider $W_{\varepsilon,d}$ the solution of

$$\begin{cases} \mathrm{div}(a_{\varepsilon,d} \nabla W_{\varepsilon,d}) = 0 & \text{in } \Omega , \\ W_{\varepsilon,d} = \Phi_b & \text{on } \partial\Omega , \end{cases} \tag{6}$$

and compare it to the unperturbed solution W_ε of

$$\begin{cases} \mathrm{div}(a_\varepsilon \nabla W_\varepsilon) = 0 & \text{in } \Omega , \\ W_\varepsilon = \Phi_b & \text{on } \partial\Omega . \end{cases} \tag{7}$$

The signature on the boundary of the defect is characterised by the quadratic form $\mathscr{R}_\varepsilon : H^{1/2}(\partial\Omega) \to \mathbb{R}$ defined by

$$\mathscr{R}_\varepsilon(\Phi_b) = \int_{\partial\Omega} a_\varepsilon \frac{\partial}{\partial n} (W_{\varepsilon,d} - W_\varepsilon)(s)\Phi_b(s)ds.$$

We show in Theorem 4 that at first order, for $\Phi_b \in C^1(\bar{\Omega})$, it is of the form

$$\mathscr{R}_\varepsilon(\Phi_b) = |G_\varepsilon| \int_\Omega M \nabla W_* \cdot \nabla W_* \, d\mu + o\left(|G_\varepsilon|\right),$$

where W_* is the solution of the homogenized problem (13) associated with (7). This can be seen as an extension of [13] to (7). At first order, the signature of the defect is similar to what would be observed if defect was introduced in the effective medium, instead of the homogeneous substrate: the only difference is the formula of polarisation tensor M. This is what was observed in [6] for finite conductivities.

Theorem 4 has another interpretation, probably of equal if not greater importance for applications. Theorem 4, compared with the main result of [1] or [6] shows that impurities in the substrate do not affect the overall properties of the composite material more than impurities would affect a regular composite, provided these impurities are located in Ω_ε, that is, not too close to the highly conducting fibres but not necessarily at a distance proportional to the size of the microstructure. As far as the authors are aware, this is the first result of this nature. If such highly contrasted structures are used to manufacture meta-materials, as it is suggested in [7], this result is of practical importance.

Another very related question is the regularity of the solution of a problem similar to (4), where cavities replace inclusions. This problem shows a similar non-standard effective behaviour, the celebrated "strange term" of Cioranescu and Murat [17]. This will be the subject of a forthcoming paper.

Our paper is structured as follows. In Section 2, we state our main results concerning the regularity of (4) away from the highly conductive fibres. First, we consider for $F \in L^2(\Omega)$, the solution $U_\varepsilon \in H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div}(a_\varepsilon \nabla U_\varepsilon) = F & \text{in } \Omega \,, \\ U_\varepsilon = 0 & \text{on } \partial\Omega \,. \end{cases} \tag{8}$$

Our result concerning problem (8) is Theorem 1. We then turn to the boundary value problem (7), and obtain an estimate for W_ε in Theorem 2. To highlight the fact that excluding a buffer zone around the highly conducting fibres is necessary, for any $p > 2$ we provide an explosive lower bound for the $W^{1,p}$ norm of W_ε in Corollary 2 (see also Remark 3). This result is a corollary of the homogenization result given by Theorem 3.

Our approach relies on the fact that one can perform a partial Fourier series decomposition of U_ε and W_ε in x_3. In Section 3 we prove Theorem 1 and Theorem 2 starting from the regularity of the Fourier coefficients. Note that translating the interior regularity results from U_ε to W_ε is not obvious. A natural idea is to study $\tilde{W}_\varepsilon = \eta W_\varepsilon$, where η is a cut-off function. Provided the cut-off function is chosen carefully, this leads to an interior problem in $H_0^1(\Omega)$,

$$-\operatorname{div}(a_\varepsilon \nabla \tilde{W}_\varepsilon) = -2a_\varepsilon \nabla W_\varepsilon \cdot \nabla \eta - a_\varepsilon \Delta\eta W_\varepsilon \text{ in } \Omega.$$

Let us focus on the first right-hand side term. Remembering that the only bound uniform in ε is the energy bound, $\int_\Omega a_\varepsilon |\nabla W_\varepsilon|^2 \leq C\|\Phi_b\|_{C^1(\bar{\Omega})}$, since ∇W_ε is not bounded uniformly in ε in any $L^p(\Omega)$ for $p > 2$, we are then left with a problem of the form

$$-\operatorname{div}(a_\varepsilon \nabla \hat{W}_\varepsilon) = -a_\varepsilon f, \text{ in } H_0^1(\Omega) \text{ with } \int_\Omega a_\varepsilon f^2 dx < \infty.$$

Unfortunately, without additional information on f, this is not enough to guarantee that $\hat{W}_\varepsilon$ is bounded in $L^\infty(\Omega)$. In Proposition 4, we show that the best one can hope for in this case is $\varepsilon\|\hat{W}_\varepsilon\|_{L^\infty} < C\|\sqrt{a_\varepsilon}f\|_{L^2}$.

In Section 4, we show how these results can be used to obtain a representation formula for inclusions of small measure located away from the fibres. This part was the initial motivation of this work. Because outside of the highly conducting fibres the substrate is homogeneous, we provide a self-contained proof.

We prove the homogenization Theorem 3 in Section 5, and finally turn to more technical results. Section 6 is devoted to the proof of Lemma 1, a supremum estimate of de Giorgi-Moser-Nash type adapted to our problem. The proof of this Lemma uses a Poincaré-Sobolev inequality proved in Lemma 4. Section 7 contains the proof of the regularity results for the Fourier coefficients associated with U_ε. Section 8 is the counterpart of Section 7, for W_ε. Several intermediate technical results are proved in Appendix A.

2 Main interior regularity estimates

The main part of this paper is devoted to the derivation of interior regularity estimates for problem (4), outside of the highly conducting rods.

For a fixed $\omega_0 \Subset \omega_1 \Subset \omega$, define ω_τ^ε by

$$\omega_\tau^\varepsilon = \{x \in \omega_1 : \operatorname{dist}(x, D_\varepsilon) \geq \varepsilon\tau\}. \tag{9}$$

Our result concerning problem (8) is the following.

Theorem 1 *For $\kappa > 0$, $\tau > \kappa\,\varepsilon^{\frac{1-\eta}{2(1+\eta)}}$ with $\eta \in \left(\frac{3}{4}, 1\right)$, and $0 < \nu < 2\left(\eta - \frac{3}{4}\right)$, the solution U_ε of (8) satisfies*

$$\|U_\varepsilon - V\|_{C^{1,\nu}(\omega_\tau^\varepsilon \times (-L,L))} \leq C(\kappa, \eta, \nu)\|F\|_{L^2(\Omega)},$$

where $V \in H_0^1(\Omega)$ is the solution of

$$-\Delta V = F.$$

This result is proved in Section 3.

Remark 1 It should be noted that, generically, V may not belong to any $W^{1,p}(\Omega)$ for $p > 6$. The above result asserts that the difference $U_\varepsilon - V$ enjoys a better regularity in most of the domain since $|\omega_1 \setminus \omega_\tau^\varepsilon| \to 0$ as $\varepsilon \to 0$.

We use the notation C for various constants in the paper which are always independent of ε. When appropriate, we highlight the dependence on the parameters appearing in the statements of the results by writing $C(a, b, \ldots)$.

As a direct consequence of the above theorem, we have

Corollary 1 *Under the assumptions of Theorem 1 we have*

$$\|U_\varepsilon\|_{L^\infty(\omega_\tau^\varepsilon \times (-L,L))} + \|\nabla U_\varepsilon\|_{L^6(\omega_\tau^\varepsilon \times (-L,L))} \leq C(\kappa, \eta)\|F\|_{L^2(\Omega)}. \tag{10}$$

Next, we derive an estimate for the solution W_ε of (7), given by the following proposition.

Theorem 2 *Let $\Omega_\tau^\varepsilon = \omega_\tau^\varepsilon \times (-l, l)$, with $l < L$. For $\kappa > 0$, $\tau > \kappa\,\varepsilon^{\frac{1-\eta}{2(1+\eta)}}$ with $\eta \in \left(\frac{3}{4}, 1\right)$, and $0 < \nu < 2\left(\eta - \frac{3}{4}\right)$, the solution W_ε of (7) satisfies*

$$\|W_\varepsilon\|_{C^{1,\nu}(\Omega_\tau^\varepsilon)} \leq C(\kappa, \eta, \nu, l)\,\|\Phi_b\|_{C^1(\bar{\Omega})}.$$

This result is proved in Section 3.

Remark 2 Once again, as $\varepsilon \to 0$, we have $|\omega_1 \setminus \omega_\tau^\varepsilon| = o(1)$, therefore the solution of (8) enjoys a uniform $C^{1,\nu}$-bound in almost the whole domain.

It is natural to ask whether a uniform global $C^{1,\nu}$-bound exists for Ω. To answer this question, first we consider the limit homogenized problem corresponding to (7). To state our homogenization result, we need to introduce a capacity function. The function c_ε^σ is defined, for some $\sigma \geq 1$, in Ω by

$$c_\varepsilon^\sigma(x) = \begin{cases} 0 & \text{if } r = \sqrt{(x_1 - m\varepsilon)^2 + (x_2 - n\varepsilon)^2} \leq \varepsilon r_\varepsilon \\ \dfrac{\ln r - \ln(\varepsilon r_\varepsilon)}{\ln(\varepsilon^\sigma/2) - \ln(\varepsilon r_\varepsilon)} & \text{if } r \in (\varepsilon r_\varepsilon, \varepsilon^\sigma/2), \quad \text{for } (m,n) \in I_\varepsilon, \\ 1 & \text{elsewhere.} \end{cases} \tag{11}$$

In fact, we choose σ so that $\varepsilon^\sigma > \varepsilon\tau$. Note that c_ε^σ does not depend on the variable x_3, is periodic of period $[-\varepsilon/2, \varepsilon/2]^3$ in Ω_0, and is equal to 1 outside of Ω_0. This capacity function has been ubiquitous in the derivation of homogenization results related to conductivities of the form (1) since its introduction in [17].

We have the following homogenization result, proved in Section 5. Note that a_ε is only reinforced in the cylinder $\Omega_0 = \omega_0 \times (-L, L)$.

Theorem 3 *Assume in addition to (2), (3), that*

$$\lim_{\varepsilon \to 0} \alpha_\varepsilon \, \pi r_\varepsilon^2 = \kappa \in [\kappa_-, \kappa_+], \quad \lim_{\varepsilon \to 0} \frac{2\pi}{\varepsilon^2 \, |\ln r_\varepsilon|} = \gamma \in [\gamma_-, \gamma_+]. \tag{12}$$

Let $\Phi_b \in C^1(\bar{\Omega})$. Then, the solution W_ε converges weakly in $H^1(\Omega)$ to the unique solution W_* of the coupled system

$$\begin{cases} -\Delta W_* + \gamma \, (W_* - V_*) \, \mathbf{1}_{\Omega_0} = 0 \text{ in } \Omega \\ -\kappa \, \partial_{33}^2 V_* + \gamma \, (V_* - W_*) = 0 \text{ in } \Omega_0 = \omega_0 \times (-L, L) \\ W_* = \Phi_b \text{ on } \partial\Omega \\ V_*(\cdot, \pm L) = \Phi_b(\cdot, \pm L) \text{ in } \omega_0. \end{cases} \tag{13}$$

The pair (W_*, V_*) satisfies $W_* \in C^{0,\alpha}(\bar{\Omega}) \cap W_{\text{loc}}^{2,p}(\Omega)$ and $V_* \in C^{0,\alpha}(\bar{\Omega}_0) \cap C_{\text{loc}}^\infty(\Omega_0)$ for some $\alpha \in (0,1)$ and for any $p > 2$.

Moreover, the following corrector result holds:

$$\lim_{\varepsilon \to 0} \int_\Omega a_\varepsilon \big| \nabla W_\varepsilon - \nabla c_\varepsilon^\sigma \, (W_* - V_*) - c_\varepsilon^\sigma \nabla W_* - (1 - c_\varepsilon^\sigma) \, \partial_3 V_* \, e_3 \big|^2 \, dx = 0. \tag{14}$$

Because of the non-local nature of the problem it solves, it is not clear that W_* is analytic inside Ω_0. The following proposition shows that it enjoys partial analyticity, that is, with respect to the x_3 variable.

Proposition 1 *Under the hypothesis of Theorem 3, the solutions W_* and V_* of (13) are analytic with respect to the x_3 variable in Ω_0.*

This result is proved in Appendix A.1. We are now in position to answer the global regularity question. Theorem 3 and Proposition 1 imply that one cannot expect an ε independent bound for the sequence W_ε in a space better than $H^1(\Omega)$ in general, as the following Corollary shows.

Corollary 2 *Let $\Phi_b \in C^2(\bar{\Omega})$ be such that the sets*

$$S_\pm(\Phi_b) = \{x' \in \omega_0 \text{ such that } \Delta_2 \Phi_b(\cdot, \pm L) = 0\}, \quad \text{where } \Delta_2 = \partial_{11}^2 + \partial_{22}^2,$$

have an intersection of zero measure. Then, the sequence W_ε is unbounded in $W_{\text{loc}}^{1,p}(\Omega_0)$ for any $p > 2$. More precisely, for any non empty open set $\Omega' \subset \Omega_0$, there exists a positive constant $C(\Omega', \Phi_b)$ independent of ε such that for any $\rho \in (0,1)$ and $p > 2$,

$$\liminf_{\varepsilon \to 0} \left(r_\varepsilon^{2\rho(1-2/p)} \|\nabla W_\varepsilon\|_{L^p(\Omega' \setminus Q_\varepsilon)}^2 \right) \geq (1-\rho) C\left(\Omega', \Phi_b\right).$$

Proof Working with a subsequence if necessary, we can assume that (12) holds.

Fix $\rho \in (0, 1)$ and let $\mathcal{O}_\varepsilon$ be the open subset of Ω_0 defined by

$$\mathcal{O}_\varepsilon = \left\{ x \in \omega_0 : 0 < \mathrm{dist}\,(x, D_\varepsilon) < \varepsilon r_\varepsilon^\rho \right\} \times (-L, L), \tag{15}$$

By the definition (11) of c_ε^σ combined with (12), and by the periodicity of c_ε^σ and $\mathbf{1}_{\mathcal{O}_\varepsilon}$ in Ω_0, we have

$$1_{\mathcal{O}_\varepsilon}\,|\nabla c_\varepsilon^\sigma|^2 \;\rightharpoonup\; \lim_{\varepsilon \to 0} \left(\frac{2\pi}{\varepsilon^2} \int_{\varepsilon r_\varepsilon}^{\varepsilon r_\varepsilon^\rho} \frac{dr}{r\,\ln^2(2\varepsilon^{1-\sigma}\,r_\varepsilon)} \right) = \gamma(1 - \rho) \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega_0),$$

which by virtue of the continuity of $V_* - W_*$ implies that

$$1_{\mathcal{O}_\varepsilon}\,|\nabla c_\varepsilon^\sigma|^2\,(V_* - W_*)^2\,dx \;\rightharpoonup\; \gamma(1 - \rho)\,(V_* - W_*)^2 \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega_0). \tag{16}$$

Let us show that this last term does not cancel on any non-empty open subset $\Omega' \subset \Omega_0$. By contradiction, suppose that the continuous function $V_* - W_* \equiv 0$ on a ball $B_R(p^0)$, centred in $p^0 = (p_1^0, p_2^0, p_3^0)$ and of radius $R > 0$, such that $B_R(p_0) \subset \Omega_0$. This implies that for example that for any $x' \in C_R$, with $C_R = (p_1^0 - R/2, p_1^0 + R/2) \times (p_2^0 - R/2, p_2^0 + R/2)$, the function $W_*(x', \cdot) - V_*(x', \cdot)$ is identically zero on $(p_3^0 - \sqrt{3}R/2, p_3^0 + \sqrt{3}R/2)$. As we have shown in Proposition 1, this function is analytic on $(-L, L)$ therefore $W_* = V_*$ on $C_R \times (-L, L)$. The system (13) then shows that $\partial_{33}^2 V_* = \Delta V_* = \Delta W_* = 0$ in $C_R \times (-L, L)$. Hence, taking into account the boundary conditions of (13) we would have

$$W_*(x) = V_*(x) = \frac{L + x_3}{2L}\,\Phi_b(x', L) + \frac{L - x_3}{2L}\,\Phi_b(x', -L) \quad \text{for any } x \in C_R \times (-L, L).$$

Since W_* is harmonic in $C_R \times (-L, L)$, it follows that both functions $\Phi_b(\cdot, \pm L)$ are harmonic in C_R, therefore $|S_+(\Phi_b) \cap S_-(\Phi_b)| \geq |C_R| > 0$ which contradicts the hypothesis.

Next, since $\nabla W_*, \partial_3 V_* \in L^2(\Omega_0)$ and $0 \leq c_\varepsilon^\sigma \leq 1$ in Ω_0, we have

$$\lim_{\varepsilon \to 0} \int_{\mathcal{O}_\varepsilon} \left| c_\varepsilon^\sigma \nabla W_* + (1 - c_\varepsilon^\sigma)\,\partial_3 V_*\,e_3 \right|^2 dx = 0.$$

Hence, by the convergence (14) and the fact that $a_\varepsilon \equiv 1$ in $\mathcal{O}_\varepsilon$, we deduce that

$$1_{\mathcal{O}_\varepsilon} \left| \nabla W_\varepsilon - \nabla c_\varepsilon^\sigma\,(W_* - V_*) \right|^2 \;\longrightarrow\; 0 \quad \text{strongly in } L^1_{\mathrm{loc}}(\Omega_0).$$

This combined with convergence (16) implies that

$$1_{\mathcal{O}_\varepsilon}|\nabla W_\varepsilon|^2 \;\rightharpoonup\; \gamma(1 - \rho)(V_* - W_*)^2 \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega_0). \tag{17}$$

To proceed, fix $\Omega'' \Subset \Omega' \subset \Omega_0$ and select a smooth cut-off function $\zeta \in C_c^\infty(\Omega')$ such that $0 \leq \zeta \leq 1$ and $\zeta \equiv 1$ in Ω''. Note that, for some $C_1 > 1$

$$C_1^{-1}\,r_\varepsilon^{2\rho} \leq |\mathcal{O}_\varepsilon| \leq C_1\,r_\varepsilon^{2\rho}.$$

Thus, (17) shows

$$\int_{\Omega''} \gamma(1 - \rho)(V_* - W_*)^2\,dx \leq \int_{\Omega_0} \gamma(1 - \rho)(V_* - W_*)^2\,\zeta\,dx$$

$$= \lim_{\varepsilon \to 0} \int_{\mathcal{O}_\varepsilon} |\nabla W_\varepsilon|^2\,\zeta\,dx$$

$$\leq \liminf_{\varepsilon \to 0} \int_{\mathcal{O}_\varepsilon \cap \Omega'} |\nabla W_\varepsilon|^2\,dx$$

$$\leq \liminf_{\varepsilon \to 0} \left(|\mathcal{O}_\varepsilon|^{1 - 2/p} \|\nabla W_\varepsilon\|^2_{L^p(\Omega' \setminus Q_\varepsilon)} \right)$$

$$\leq C_1 \liminf_{\varepsilon \to 0} \left(r_\varepsilon^{2\rho(1 - 2/p)} \|\nabla W_\varepsilon\|^2_{L^p(\Omega' \setminus Q_\varepsilon)} \right).$$

Since this is valid for all $\Omega'' \Subset \Omega'$, we conclude the proof. □

Remark 3 Note that Corollary 2 shows that gradient blow-up occurs outside of the highly conducting rods. An variant of the proof shows that one can obtain the following estimate

$$\liminf_{\varepsilon \to 0} \left(r_\varepsilon^{2\rho(1-2/p)} \|\nabla W_\varepsilon\|_{L^p(\Omega' \setminus \hat{Q}_\varepsilon^{\rho'})}^2 \right) \geq (\rho' - \rho) C\left(\Omega', \Phi_b\right),$$

where $0 < \rho < \rho' < 1$ and

$$\hat{Q}_\varepsilon^{\rho'} = \left\{ x \in \omega_0 : \operatorname{dist}(x, D_\varepsilon) \leq \varepsilon r_\varepsilon^{\rho'} \right\} \times (-L, L),$$

which shows that the blow-up is not localized on the surface of the rods.

3 Regularity estimates in two dimensions for weighted equations and three-dimensional consequences

In this section, we state the regularity results we have obtained for two-dimensional companion problems of problem (4). To highlight the sharpness of the supremum estimate provided by Lemma 1, which provides an L^∞ upper bound of the form $C(s)\varepsilon^{-s}$ for any $2 > s > 1$, we construct in Proposition 4 an example where the L^∞ norm is bounded from below by $C\epsilon^{-1}$. Then, we prove Theorem 1 and Theorem 2.

3.1 Towards the proof of Theorem 1

Let us now turn to the proof of Theorem 1. Because none of the coefficients depend on x_3, we can, as in [10], reduce the study of this three-dimensional problem to that of a two-dimensional problem by separating variables. We write

$$U_\varepsilon(x', x_3) \sim \sum_{n=1}^\infty u_{\varepsilon,n}(x') \sin\left(\frac{n\pi}{2}\left(\frac{x_3}{L} + 1\right)\right), \tag{18}$$

$$F(x', x_3) \sim \sum_{n=1}^\infty f_n(x') \sin\left(\frac{n\pi}{2}\left(\frac{x_3}{L} + 1\right)\right), \tag{19}$$

and (8) becomes

$$\begin{cases} -\operatorname{div}_2(a_\varepsilon \nabla_2 u_{\varepsilon,n}) + \frac{n^2\pi^2}{4L^2} a_\varepsilon u_{\varepsilon,n} = f_n & \text{in } \omega, \\ u_{\varepsilon,n} = 0 & \text{on } \partial\omega. \end{cases} \tag{20}$$

In the above, div_2 and ∇_2 denotes the horizontal divergence and gradient operators. We are thus led to consider, for $\lambda \geq \frac{\pi^2}{4L^2}$,

$$\begin{cases} -\operatorname{div}_2(a_\varepsilon \nabla_2 u_\varepsilon) + \lambda a_\varepsilon u_\varepsilon = f + a_\varepsilon g + \operatorname{div}_2(\mathbf{h}) & \text{in } \omega, \\ u_\varepsilon = 0 & \text{on } \partial\omega, \end{cases} \tag{21}$$

where $f \in L^2(\omega)$, $\sqrt{a_\varepsilon} g \in L^2(\omega)$ and $h \in L^\infty(\omega)^2$. Problem (21) is equivalent to Problem (20) if $g = 0, \mathbf{h} = 0$; the additional $a_\varepsilon g + \operatorname{div}_2(\mathbf{h})$ term will prove useful for the study of boundary value problem (7).

Section 7 is devoted to the proof of the following result.

Proposition 2 *Assume that* $\lambda \geq \lambda_0 > 0$. *For* $\kappa > 0$, $\eta \in (0,1)$, *and* $\tau > \kappa \, \varepsilon^{\frac{1-\eta}{2(1+\eta)}}$ *the solution* u_ε *to* (21) *enjoys the following bound*

$$\|u_\varepsilon - v\|_{L^\infty(\omega_\tau^\varepsilon)} + \|\nabla_2 u_\varepsilon - \nabla_2 v\|_{L^\infty(\omega_\tau^\varepsilon)}$$
$$\leq \frac{C(\lambda_0, \kappa, \eta)}{\lambda^{\frac{\eta}{2}}} \left[\frac{1}{\lambda^{\frac{\eta}{2}}} \left(\|f\|_{L^2(\omega)} + \|\sqrt{a_\varepsilon}\, g\|_{L^2(\omega)} + \|\mathbf{h}\|_{L^\infty(\omega)^2} \right) + \|u_\varepsilon\|_{L^\infty(\omega)} \right],$$

where $v \in H_0^1(\omega)$ *is the solution of*

$$-\Delta v + \lambda v = f + g + \mathrm{div}_2(\mathbf{h}).$$

Furthermore, for $0 < \nu < \eta$,

$$[\nabla_2 u_\varepsilon - \nabla_2 v]_{C^\nu(\omega_\tau^\varepsilon)}$$
$$\leq \frac{C(\lambda_0, \kappa, \eta, \nu)}{\lambda^{\frac{\eta}{2} - \frac{\nu}{2}}} \left[\frac{1}{\lambda^{\frac{\eta}{2}}} \left(\|f\|_{L^2(\omega)} + \|\sqrt{a_\varepsilon}\, g\|_{L^2(\omega)} + \|\mathbf{h}\|_{L^\infty(\omega)^2} \right) + \|u_\varepsilon\|_{L^\infty(\omega)} \right].$$

Throughout the paper $[\cdot]_{C^\nu}$ will be used to denote the ν-Hölder semi-norm. The second ingredient for the proof is a supremum estimate for u_ε.

Lemma 1 *Assume that conditions* (1),(2) *and* (3) *hold. Assume that* $\varphi_\varepsilon \in H_0^1(\omega)$ *satisfies,*

$$- \mathrm{div}\left(a_\varepsilon \nabla \varphi_\varepsilon \right) + \lambda a_\varepsilon \varphi_\varepsilon = a_\varepsilon f_\varepsilon + \mathrm{div}\left(a_\varepsilon \mathbf{h} \right). \tag{22}$$

Then, for any $1 < \alpha < 2$, $0 < \beta < 1 - \alpha/2$, *we have*

$$\|\varphi_\varepsilon\|_{L^\infty(\omega)} \leq \frac{C(\alpha, \beta)}{\varepsilon^\alpha \lambda^\beta} \left(\|\sqrt{a_\varepsilon} f\|_{L^2(\omega)} + \varepsilon^{\frac{\alpha+1}{2}} \|\sqrt{a_\varepsilon} h\|_{L^\infty(\omega)} \right).$$

This variation on the standard de Giorgi-Moser-Nash estimates is proved in Section 6. As a direct consequence we have the following proposition.

Proposition 3 *Assume that* $\lambda \geq \lambda_0 > 0$, $g = 0$ *and* $\mathbf{h} = 0$. *We have*

$$\|u_\varepsilon\|_{L^\infty(\omega)} \leq \frac{C(\lambda_0, q)}{\lambda^{1/q}} \|f\|_{L^2(\omega)} \tag{23}$$

for any $q > 2$.

Note that Lemma 1 does not show that $\|\varphi_\varepsilon\|_{L^\infty(\omega)}$ is bounded independently of ε. The following counter-example documents the sharpness of our estimate, as it shows that it must be at least $O(\varepsilon^{-1})$.

Proposition 4 *Assume that conditions* (1), (2) *and* (3) *hold. Let* $\varphi_\varepsilon \in H_0^1(\omega)$ *be the solution of* (22). *Let* $\mathbf{1}_{D_{0,0,\varepsilon}}$ *be the indicator function of the disk* $D_{0,0,\varepsilon} \subset \mathbb{R}^2$ *centred at the origin and of radius* $\varepsilon r_\varepsilon$. *Set* $f_\varepsilon = \varepsilon^{-1} \mathbf{1}_{D_{0,0,\varepsilon}}$ *and* $\mathbf{h} \equiv 0$. *Then, we have*

$$\|\sqrt{a_\varepsilon} f_\varepsilon\|_{L^2(\omega)} \leq \sqrt{\kappa_+}, \quad \text{and} \quad \|\varphi_\varepsilon\|_{L^\infty(\omega)} \geq \frac{1}{\varepsilon} \frac{\kappa_-}{\kappa_+} \frac{\kappa_-}{\gamma_+ + \lambda \kappa_+ + o(1)}, \tag{24}$$

where $\lim_{\varepsilon \to 0} o(1) = 0$.

Proof of Theorem 1. Fix some $l \in (0, L)$ and let $\Omega_\tau^\varepsilon = \omega_\tau^\varepsilon \times (-l, l)$. It suffices to provide the required estimates in Ω_τ^ε with a constant that is independent of l.

Together with (18) and (19) we consider the Fourier decomposition

$$V(x', x_3) \sim \sum_{n=1}^\infty v_n(x') \sin\left(\frac{n\pi}{2}\left(\frac{x_3}{L} + 1\right)\right).$$

Note that $U_\varepsilon - V$ is harmonic, thus regular, in $\Omega \setminus Q_\varepsilon$. Therefore, by Dirichlet's theorem on the convergence of Fourier series, the Fourier expansions of $U_\varepsilon - V$ and $\nabla(U_\varepsilon - V)$ converge pointwise to $U_\varepsilon - V$ and $\nabla(U_\varepsilon - V)$ in Ω_τ^ε. Proposition 2 and Proposition 3 combined with the choice $q = \frac{2}{\eta}$ show that

$$\|u_{\varepsilon,n} - v_n\|_{L^\infty(\omega_\tau^\varepsilon)} + \|\nabla_2 u_{\varepsilon,n} - \nabla_2 v_n\|_{L^\infty(\omega_\tau^\varepsilon)} \leq \frac{C(\kappa, \eta)}{n^{2\eta}} \|f_n\|_{L^2(\omega)}, \tag{25}$$

$$[\nabla_2 u_{\varepsilon,n} - \nabla_2 v_n]_{C^\nu(\omega_\tau^\varepsilon)} \leq \frac{C(\kappa, \eta)}{n^{2\eta-\nu}} \|f_n\|_{L^2(\omega)}, \tag{26}$$

with $0 < \nu < \eta < 1$. For $\eta > \frac{1}{4}$, (25) shows that the sums

$$I_1(x', x_3) = \sum_{n=1}^\infty (u_{\varepsilon,n}(x') - v_n(x')) \sin\left(\frac{n\pi}{2}\left(\frac{x_3}{L} + 1\right)\right),$$

$$I_2(x', x_3) = \sum_{n=1}^\infty (\nabla_2 u_{\varepsilon,n}(x') - \nabla_2 v_n(x')) \sin\left(\frac{n\pi}{2}\left(\frac{x_3}{L} + 1\right)\right)$$

are absolutely convergent,

$$\|I_1\|_{L^\infty(\Omega_\tau^\varepsilon)} + \|I_2\|_{L^\infty(\Omega_\tau^\varepsilon)} \leq C \sum_{n=1}^\infty \frac{1}{n^{2\eta}} \|f_n\|_{L^2(\omega)}$$

$$\leq C\|F\|_{L^2(\Omega)} \left(\sum_{n=1}^\infty \frac{1}{n^{4\eta}}\right)^{\frac{1}{2}} \leq \frac{C}{\eta - \frac{1}{4}} \|F\|_{L^2(\Omega)}.$$

This shows that

$$\|U_\varepsilon - V\|_{L^\infty(\Omega_\tau^\varepsilon)} + \|\nabla_2 U_\varepsilon - \nabla_2 V\|_{L^\infty(\Omega_\tau^\varepsilon)} \leq C\|F\|_{L^2(\Omega)}.$$

We turn to the Hölder estimates of $\nabla_2 U_\varepsilon - \nabla_2 V$. By (26) with $\eta > \frac{\nu}{2} + \frac{1}{4}$ we have,

$$|I_2(x', x_3) - I_2(y', x_3)| \leq \sum_{n=1}^\infty |(\nabla_2 u_{\varepsilon,n}(x') - \nabla_2 v_n(x')) - (\nabla_2 u_{\varepsilon,n}(y') - \nabla_2 v_n(y'))|$$

$$\leq C \sum_{n=1}^\infty \frac{1}{n^{2\eta-\nu}} \|f_n\|_{L^2(\omega)} |x' - y'|^\nu$$

$$\leq \frac{C}{\eta - \frac{\nu}{2} - \frac{1}{4}} \|F\|_{L^2(\Omega)} |x' - y'|^\nu.$$

By (25) but with $\eta > \frac{\nu}{2} + \frac{1}{4}$ we obtain,

$$|I_2(y', x_3) - I_2(y', y_3)|$$
$$\leq \sum_{n=1}^{\infty} |\nabla_2 u_{\varepsilon,n}(y') - \nabla_2 v_n(y')| \left| \sin\left(\frac{n\pi}{2}\left(\frac{x_3}{L} + 1 \right) \right) - \sin\left(\frac{n\pi}{2}\left(\frac{y_3}{L} + 1 \right) \right) \right|$$
$$\leq C \sum_{n=1}^{\infty} \frac{1}{n^{2\eta}} \|f_n\|_{L^2} |nx_3 - ny_3|^{\nu}$$
$$\leq \frac{C}{\eta - \frac{\nu}{2} - \frac{1}{4}} \|F\|_{L^2(\Omega)} |x_3 - y_3|^{\nu}.$$

From these last two estimates, we deduce

$$[\nabla_2 U_\varepsilon - \nabla_2 V]_{C^\nu(\Omega_\tau^\varepsilon)} \leq C\|F\|_{L^2(\Omega)}.$$

Next, we estimate $\partial_{x_3} U_\varepsilon - \partial_{x_3} V$. Choosing $\eta > \frac{3}{4}$ in (25) shows that the sum

$$I_3 = \sum_{n=1}^{\infty} n \left(u_{\varepsilon,n}(x') - v_n(x') \right) \cos\left(\frac{n\pi}{2}\left(\frac{x_3}{L} + 1 \right) \right)$$

converges absolutely with bound

$$\|I_3\|_{L^\infty(\Omega_\tau^\varepsilon)} \leq C \sum_{n=1}^{\infty} \frac{1}{n^{2\eta-1}} \|f_n\|_{L^2(\omega)} \leq \frac{C}{\eta - \frac{3}{4}} \|F\|_{L^2(\Omega)}.$$

This proves that

$$\|\partial_{x_3} U_\varepsilon - \partial_{x_3} V\|_{L^\infty(\Omega_\tau^\varepsilon)} \leq C\|F\|_{L^2(\Omega)}.$$

For the Hölder estimate, we proceed as before. First, with $\eta > \frac{3}{4}$ in (25) we obtain

$$|I_3(x', x_3) - I_3(y', x_3)| \leq \sum_{n=1}^{\infty} n \left| (u_{\varepsilon,n}(x') - v_n(x')) - (u_{\varepsilon,n}(y') - v_n(y')) \right|$$
$$\leq C \sum_{n=1}^{\infty} \frac{1}{n^{2\eta-1}} \|f_n\|_{L^2(\omega)} |x' - y'|$$
$$\leq \frac{C}{\eta - \frac{3}{4}} \|F\|_{L^2(\Omega)} |x' - y'|.$$

Also by (25) but with $\eta > \frac{\nu}{2} + \frac{3}{4}$,

$$|I_3(y', x_3) - I_3(y', y_3)|$$
$$\leq \sum_{n=1}^{\infty} n |u_{\varepsilon,n}(y') - v_n(y')| \left| \cos\left(\frac{n\pi}{2}\left(\frac{x_3}{L} + 1 \right) \right) - \cos\left(\frac{n\pi}{2}\left(\frac{y_3}{L} + 1 \right) \right) \right|$$
$$\leq C \sum_{n=1}^{\infty} \frac{1}{n^{2\eta-1}} \|f_n\|_{L^2} |nx_3 - ny_3|^{\nu}$$
$$\leq \frac{C}{\eta - \frac{\nu}{2} - \frac{3}{4}} \|F\|_{L^2(\Omega)} |x_3 - y_3|^{\nu}.$$

We have shown

$$[\partial_{x_3} U_\varepsilon - \partial_{x_3} V]_{C^\nu(\Omega_\tau^\varepsilon)} \le C\|F\|_{L^2(\Omega)},$$

which concludes the proof. $\square$

Proof of Proposition 4. The bound on f_ε in (24) is straightforward, from the definition of a_ε, (1), and (2). Integrating (22) against φ_ε and using (2), we find

$$\int_\omega a_\varepsilon \, |\nabla \varphi_\varepsilon|^2 \, dx + \int_\omega \lambda \, a_\varepsilon \, \varphi_\varepsilon^2 \, dx = \int_\omega a_\varepsilon \, f_\varepsilon \, \varphi_\varepsilon \, dx = \int_{D_{0,0,\varepsilon}} a_\varepsilon \, \varphi_\varepsilon \, \varepsilon^{-1} dx \le \kappa_+ \, \varepsilon \, \|\varphi_\varepsilon\|_{L^\infty(\omega)} \, .$$

Let $Y = (-1/2, 1/2)^2$ be the unit period cell, and set

$$g_\varepsilon = \varepsilon^{-1}(1 - c_\varepsilon^1) \, \mathbf{1}_{\varepsilon Y} \in H_0^1(\omega), \tag{27}$$

where c_ε^1 is defined by (11) for $\sigma = 1$. An easy computation using (2) and (3), shows that

$$\int_\omega a_\varepsilon \, |\nabla g_\varepsilon|^2 \, dx + \int_\omega \lambda \, a_\varepsilon \, g_\varepsilon^2 \, dx = \int_{\varepsilon Y} \left|\nabla c_\varepsilon^1\right|^2 \varepsilon^{-2} \, dx + \int_{\varepsilon Y} \lambda \, a_\varepsilon \, (1 - c_\varepsilon^1)^2 \, \varepsilon^{-2} dx$$
$$\le \gamma_+ + \lambda \, \kappa_+ + o(1).$$

Then, the Cauchy-Schwarz inequality combined with the two previous estimates yields

$$\int_\omega a_\varepsilon \, \nabla \varphi_\varepsilon \cdot \nabla g_\varepsilon \, dx + \int_\omega a_\varepsilon \, \varphi_\varepsilon \, g_\varepsilon \, dx \le \left(\kappa_+ \, \varepsilon \, \|\varphi_\varepsilon\|_{L^\infty(\omega)}\right)^{\frac{1}{2}} \left(\gamma_+ + \lambda \, \kappa_+ + o(1)\right)^{\frac{1}{2}} \, .$$

On the other hand, integrating (22) against g_ε, we find

$$\int_\omega a_\varepsilon \, \nabla \varphi_\varepsilon \cdot \nabla g_\varepsilon \, dx + \int_\omega a_\varepsilon \, \varphi_\varepsilon \, g_\varepsilon \, dx = \int_\omega a_\varepsilon \, f_\varepsilon \, g_\varepsilon \, dx = \int_{D_{0,0,\varepsilon}} a_\varepsilon \, \varepsilon^{-2} dx \ge \kappa_-.$$

Combining these two inequalities provides the announced bound. $\square$

3.2 Towards the proof of Theorem 2

Let us now turn to the proof of Theorem 2. Our approach is to decompose the boundary data Φ_b into three parts. We first define

$$\phi_L(\cdot, x_3) = \frac{x_3}{2L} \left(\Phi_b(\cdot, L) - \Phi_b(\cdot, -L)\right) + \frac{1}{2} \left(\Phi_b(\cdot, L) + \Phi_b(\cdot, -L)\right), \tag{28}$$

which agrees with $\Phi_b(\cdot, \pm L)$ on $\bar{\omega} \times \{\pm L\}$. We now freeze the variations of ϕ_L in the x'-direction on the highly conducting fibres by decomposing ϕ_L into

$$\phi_L = \phi_1 + \phi_2,$$

where

$$\phi_1 = \phi_L c_\varepsilon^\sigma + \sum_{(m,n) \in I_\varepsilon} (1 - c_\varepsilon^\sigma) \, \mathbf{1}_{\{|(x_1, x_2) - (m,n)\varepsilon| \le \varepsilon/2\}} \, (x) \, \phi_L \, (m\varepsilon, n\varepsilon, x_3) \, , \tag{29}$$

the capacity function c_ε^σ being defined in (11). We finally define ϕ_0 to be the trace of Φ_b on the lateral boundary of the cylinder Ω,

$$\phi_0 = \Phi_b - \phi_L = \Phi_b - \phi_1 - \phi_2 \quad \text{on } \partial\Omega.$$

Note that ϕ_0 vanishes on $\bar{\omega} \times \{\pm L\}$.

The properties of ϕ_1 and ϕ_2 we will use are described by the following proposition.

Proposition 5 *The function ϕ_1 given by (29)*

- *belongs to $C^\infty([-L, L]; C^{0,1}(\bar\omega))$,*
- *is independent of x_1 and x_2 inside each connected component of Q_ε,*
- *satisfies $\partial_{33}\phi_1 = 0$ in Ω,*
- *is globally Lipschitz,*

$$\|\phi_1\|_{C^{0,1}(\bar\Omega)} \leq C\|\phi_L\|_{C^1(\bar\Omega)}.$$

The function $\phi_2 = \phi_L - \phi_1$ where ϕ_L is defined in (28)

- *is globally Lipschitz,*

$$\|\phi_2\|_{C^{0,1}(\bar\Omega)} \leq C\|\phi_L\|_{C^1(\bar\Omega)},$$

- *is supported by*

$$\omega_{R_\varepsilon} = \left\{ x \in \Omega_0 \ s.t. \ \min_{(m,n)\in I_\varepsilon} |(x_1, x_2) - (m\varepsilon, n\varepsilon)| \leq \varepsilon^\sigma \right\},$$

- *and satisfies*

$$\|\phi_2\|_{L^\infty(\Omega)} \leq C\varepsilon^{\sigma+2}\|\phi_L\|_{C^1(\bar\Omega)}.$$

The proof is straightforward and given in Appendix A.2.

Proof of Theorem 2. We will study three boundary problems separately. For $i = 0, 1, 2$, we introduce $W_{\varepsilon,i}$, the solution of

$$\begin{cases} \operatorname{div}(a_\varepsilon \nabla W_{\varepsilon,i}) = 0 & \text{in } \Omega, \\ W_{\varepsilon,i} = \phi_i & \text{on } \partial\Omega. \end{cases} \tag{30}$$

By linearity, $W_\varepsilon = W_{\varepsilon,0} + W_{\varepsilon,1} + W_{\varepsilon,2}$.

Consider $\tilde{W}_{\varepsilon,0} = \zeta_0 W_{\varepsilon,0}$ where ζ_0 is a cut-off function $\zeta_0 \in C_c^\infty(\omega_3)$, such that

$$0 \leq \zeta_0 \leq 1 \text{ and } \zeta_0 \equiv 1 \text{ in } \omega_2, \tag{31}$$

with $\omega_1 \Subset \omega_2 \Subset \omega_3 \Subset \omega$. Since $\zeta_0 \equiv 1$ on ω_0, $\tilde{W}_{\varepsilon,0} \in H_0^1(\Omega)$ satisfies

$$-\operatorname{div}(a_\varepsilon \nabla \tilde{W}_{\varepsilon,0}) = -2\nabla\zeta_0 \cdot \nabla W_{\varepsilon,0} - (\Delta\zeta_0) W_{\varepsilon,0} \text{ in } \Omega. \tag{32}$$

The maximum principle shows that $\|W_{\varepsilon,0}\|_{L^\infty(\Omega)} \leq \|\phi_0\|_{C^0(\bar\Omega)}$. Furthermore, on $\Omega_0^c = (\omega \setminus \omega_0) \times (-L, L)$, $u_\varepsilon = W_{\varepsilon,0}$ is the solution of

$$\begin{aligned} \Delta u_\varepsilon &= 0 \text{ in } \Omega_0^c, \\ u_\varepsilon &= 0 \text{ on } (\omega \setminus \omega_0) \times \{-L, L\}, \\ u_\varepsilon &= W_{\varepsilon,0} \text{ on } \partial(\omega \setminus \omega_0) \times (-L, L). \end{aligned}$$

Classical regularity estimates then show that

$$\|W_{\varepsilon,0}\|_{C^2(\overline{(\omega_3\setminus\omega_2)\times(-L,L)})} \leq C(\omega_2, \omega_3, l)\,\|W_{\varepsilon,0}\|_{L^\infty(\Omega)}. \tag{33}$$

Thus, the right-hand side of (32) is in $C^1(\bar\Omega)$, and we can apply Theorem 1 to conclude that for $\tau > \kappa\varepsilon^{\frac{1-\eta}{2(1+\eta)}}$ with $\eta \in \left(\frac{3}{4}, 1\right)$, and $0 < \nu < 2\left(\eta - \frac{3}{4}\right)$, we have

$$\left\|\tilde{W}_{\varepsilon,0} - \tilde{V}_0\right\|_{C^{1,\nu}(\Omega_\tau^\varepsilon)} \leq C\,\|W_{\varepsilon,0}\|_{L^\infty(\Omega)},$$

where $\tilde{V}_0 \in H_0^1(\Omega)$ is the solution of

$$-\Delta\tilde{V}_0 = -2\nabla\zeta_0 \cdot \nabla W_{\varepsilon,0} - (\Delta\zeta_0) W_{\varepsilon,0} \text{ in } \Omega.$$

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Since the right-hand side is $C^1(\bar{\Omega})$, $\tilde{V}_0$ enjoys interior regularity, and we have obtained

$$\|W_{\varepsilon,0}\|_{C^{1,\nu}(\Omega_\tau^\varepsilon)} \leq C \|\Phi_b\|_{C^0(\bar{\Omega})}. \tag{34}$$

Next, we consider $W_{\varepsilon,1}$. We note that for any $v \in H_0^1(\Omega)$,

$$\begin{aligned}
\int_\Omega a_\varepsilon \nabla (W_{\varepsilon,1} - \phi_1) \cdot \nabla v &= -\int_\Omega a_\varepsilon \nabla \phi_1 \cdot \nabla v, \\
&= \int_{Q_\varepsilon} (1 - a_\varepsilon) \nabla \phi_1 \cdot \nabla v - \int_\Omega \nabla \phi_1 \cdot \nabla v, \\
&= \int_{Q_\varepsilon} (1 - a_\varepsilon) \partial_3 \phi_1 \cdot \partial_3 v + \int_\Omega \Delta \phi_1 v, \\
&= \int_\Omega \Delta \phi_1 v.
\end{aligned}$$

We used first that ϕ_1 is independent of x_1 and x_2 in each connected component of Q_ε, and then that it satisfies $\partial_{33}\phi_1 = 0$ (see Proposition 5). In other words, introducing $\tilde{W}_{\varepsilon,1} = W_{\varepsilon,1} - \phi_1$, we have

$$-\operatorname{div}\left(a_\varepsilon \nabla \tilde{W}_{\varepsilon,1}\right) = \Delta \phi_1 \in \Omega. \tag{35}$$

Following the strategy used for the interior source problem, we thus introduce $\tilde{V}_1 \in H_0^1(\Omega)$, the solution of

$$-\Delta \tilde{V}_1 = \Delta \phi_1 \text{ in } \Omega,$$

In Section 8, we will show in Proposition 9 that, for $\tau > \kappa \varepsilon^{\frac{1-\eta}{2(1+\eta)}}$, $\frac{1}{2} < \eta < 1$, $0 < \nu < 2(\eta - \frac{1}{2})$ that

$$\left\|\tilde{W}_{\varepsilon,1} - \tilde{V}_1\right\|_{C^{1,\nu}(\Omega_\tau^\varepsilon)} \leq C \|\Phi_b\|_{C^1(\bar{\Omega})}.$$

On the other hand, as $\tilde{V}_1 + \phi_1$ is harmonic in Ω, we have

$$\left\|\tilde{V}_1 + \phi_1\right\|_{C^{1,\nu}(\Omega_\tau^\varepsilon)} \leq C \|\Phi_b\|_{L^\infty(\Omega)}.$$

As $W_{\varepsilon,1} = (\tilde{W}_{\varepsilon,1} - \tilde{V}_1) + (\tilde{V}_1 + \phi_1)$, we have thus shown that

$$\|W_{\varepsilon,1}\|_{C^{1,\nu}(\Omega_\tau^\varepsilon)} \leq C \|\Phi_b\|_{L^\infty(\Omega)}. \tag{36}$$

Finally, for $W_{\varepsilon,2}$, we shall consider instead the function $\tilde{W}_{\varepsilon,2} = \zeta W_{\varepsilon,2}$ where $\zeta(x) = p(x_3)\zeta_0$ and ζ_0 is the cut-off function given by (31), and $p \in C_c^\infty(-l',l')$ with $l < l' < L$ is such that

$$0 \leq p \leq 1 \text{ and } p \equiv 1 \text{ in } (-l,l). \tag{37}$$

We compute

$$-\operatorname{div}\left(a_\varepsilon \nabla \tilde{W}_{\varepsilon,2}\right) = a_\varepsilon \left(2\partial_3\zeta \partial_3 W_{\varepsilon,2} + \partial_{33}\zeta W_{\varepsilon,2}\right) + g_\varepsilon, \tag{38}$$

where g_ε has support in $(\omega_3 \setminus \omega_2) \times (-l',l')$, and is given by

$$g_\varepsilon = (\partial_{11}\zeta + \partial_{22}\zeta) W_{\varepsilon,2} + 2\partial_1\zeta \partial_1 W_{\varepsilon,2} + 2\partial_2\zeta \partial_2 W_{\varepsilon,2}.$$

By the standard maximum principle,

$$\|W_{\varepsilon,2}\|_{L^\infty(\Omega)} \leq \|\phi_2\|_{L^\infty(\Omega)}. \tag{39}$$

Just like $\tilde{W}_{\varepsilon,0}$ enjoys additional regularity away from the support of the rods as shown by (33), we have, for any $l' < l'' < L$,

$$\|W_{\varepsilon,2}\|_{C^2(\overline{(\omega_3 \backslash \omega_2) \times (-l'',l'')})} \leq C(\omega_2, \omega_3, l'') \|W_{\varepsilon,2}\|_{L^\infty(\Omega)}. \tag{40}$$

Next, notice that $\partial_{x_3} W_{\varepsilon,2}$ and $W_{\varepsilon,2}$ are both a_ε-harmonic, a standard energy estimate shows that, for any $\Omega' \Subset \Omega'' \Subset \Omega$, we have

$$\int_{\Omega'} a_\varepsilon \left|\nabla(\partial_3 W_{\varepsilon,2})\right|^2 dx \leq C \int_{\Omega''} a_\varepsilon \left|\partial_3 W_{\varepsilon,2}\right|^2 dx \leq C \int_\Omega a_\varepsilon \left|W_{\varepsilon,2}\right|^2 dx \leq C \|W_{\varepsilon,2}\|_{L^\infty(\Omega)}. \tag{41}$$

Note that the regularity estimate (40) yields in particular that

$$\|g_\varepsilon\|_{C^1(\bar\Omega)} \leq C \|\phi_2\|_{L^\infty(\Omega)}. \tag{42}$$

The companion problem to (38) is

$$-\Delta \tilde{V}_{\varepsilon,2} = 2\partial_3 \zeta \partial_3 W_{\varepsilon,2} + \partial_{33} \zeta \, W_{\varepsilon,2} + g_\varepsilon \text{ in } \Omega, \tag{43}$$

Thanks to (40), the right-hand side of (43) is in $C^1(\bar\Omega)$ uniformly in ε, and therefore $\tilde{V}_{\varepsilon,2}$ enjoys interior regularity. Proposition 10 proved in Section 8 on the regularity of the difference $\tilde{W}_{\varepsilon,2} - \tilde{V}_{\varepsilon,2}$ implies a regularity result of $W_{\varepsilon,2}$. For $\kappa > 0$, $\tau > \kappa \varepsilon^{\frac{1-\eta}{2(1+\eta)}}$ with $\eta \in \left(\frac{2}{3}, 1\right)$, and $0 < \nu < 3\left(\eta - \frac{2}{3}\right)$, the solution $W_{\varepsilon,2}$ of (30) satisfies

$$\|W_{\varepsilon,2}\|_{C^{1,\nu}(\Omega_\tau^\varepsilon)} \leq C \varepsilon^\sigma \|\Phi_b\|_{C^1(\bar\Omega)}. \tag{44}$$

The claim of the Theorem is a consequence of the three bounds (34), (36) and (44). $\square$

4 Application to structures with defects of small volume

We now consider the case when a defect of small volume is present in the medium. In that case, the conductivity of the defective medium is given by (5). We assume that the defect of support G_ε stays away from the high conductivity fibres. To fix ideas, given $0 < l < L$ and ω_1 such that $\omega_0 \Subset \omega_1 \Subset \omega$, suppose

$$G_\varepsilon \subset \Omega_\varepsilon = \left\{x \in \omega_1 : \text{dist}\,(x, D_\varepsilon) \geq \varepsilon^{17/16}\right\} \times (-l, l). \tag{45}$$

Note that the set Ω_ε grows as ε tends to zero. Furthermore, in the language of Section 2, $\Omega_\varepsilon = \Omega_\tau^\varepsilon$ for $\tau = \varepsilon^{1/16}$, $\kappa = 1$, $\eta = 7/9$. This guarantees a $C^{1,\nu}$ estimate with $0 < \nu < 1/18$.

Throughout the section, we will assume that (12) holds, namely

$$\lim_{\varepsilon \to 0} \alpha_\varepsilon \pi r_\varepsilon^2 = \kappa \in [\kappa_-, \kappa_+], \quad \lim_{\varepsilon \to 0} \frac{2\pi}{\varepsilon^2 |\ln r_\varepsilon|} = \gamma \in [\gamma_-, \gamma_+].$$

When $\varepsilon \to 0$, Theorem 3 shows that the sequence W_ε converges weakly in $H_0^1(\Omega)$ to the solution W_* of the coupled system (13) when $\Phi_b \in C^1(\bar\Omega)$.

We proceed to derive an asymptotic formula for the difference response operator (the difference of Dirichlet-to-Neumann operators on $\partial\Omega$)

$$H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$
$$\Phi_b \mapsto a_\varepsilon \left.\frac{\partial}{\partial n}(W_{\varepsilon,d} - W_\varepsilon)\right|_{\partial\Omega}.$$

As this operator can be recovered by polarisation, we limit ourselves to the study of the quadratic form

$$\mathscr{R}_\varepsilon : H^{1/2}(\partial\Omega) \to \mathbb{R}$$
$$\Phi_b \mapsto \int_{\partial\Omega} a_\varepsilon \frac{\partial}{\partial n} \left(W_{\varepsilon,d} - W_\varepsilon\right)(s)\Phi_b(s)ds. \tag{46}$$

We can now state the main result of this section.

Theorem 4 *Assume that $|G_\varepsilon| \to 0$, and $|G_\varepsilon|^{-1}\mathbf{1}_{G_\varepsilon}$ converges weakly-$*$ to a Radon measure μ on $\bar{\Omega}$. There exist a subsequence, still denoted by ε, and a matrix-valued function $M \in L^2(\Omega,\mu)^{3\times 3}$ such that the bilinear response form $\mathscr{R}_\varepsilon$ given by (46) has the following asymptotic form*

$$\mathscr{R}_\varepsilon(\Phi_b) = |G_\varepsilon| \int_\Omega M\nabla W_* \cdot \nabla W_* \, d\mu + o\left(|G_\varepsilon|\right) \text{ for any } \Phi_b \in C^1(\bar{\Omega}), \tag{47}$$

where W_* denotes the solution to the homogenized problem (13) with boundary condition Φ_b, and

$$\lim_{\varepsilon \to 0} \sup_{\|\Phi_b\|_{C^1(\bar{\Omega})} \leq 1} \frac{o(|G_\varepsilon|)}{|G_\varepsilon|} = 0.$$

In addition, the matrix-valued function M is symmetric, independent of Φ_b, and satisfies

$$(\gamma_1 - 1)\min\left(1, \frac{1}{\gamma_1}\right) \leq M(y)\xi \cdot \xi \leq (\gamma_1 - 1)\max\left(1, \frac{1}{\gamma_1}\right) \ \mu\text{-a.e.}$$

Remark 4 Due to the definition (45) of Ω_ε, the measure μ has actually support in $\bar{\omega}_1 \times [-l,l] \Subset \Omega$. This gives a sense to the integral term of (47) since $W_* \in C^1_{\text{loc}}(\Omega)$.

Remark 5 We have characterised the signature of a defect provided that the boundary condition Φ_b is in $C^1(\bar{\Omega})$. The natural space for Φ_b is $H^{1/2}(\partial\Omega)$, but we do not know if our result holds in this space as well.

Our strategy is inspired by [13,1]. The first key element is the following Lemma.

Lemma 2 *There exist a positive constant C independent of Φ_b and ε such that*

$$\|\nabla(W_\varepsilon - W_{\varepsilon,d})\|_{L^2(\Omega)} + |G_\varepsilon|^{-1/3}\|W_\varepsilon - W_{\varepsilon,d}\|_{L^2(\Omega)} \leq C\,|G_\varepsilon|^{1/2}\,\|\Phi_b\|_{C^1(\bar{\Omega})}.$$

Proof Write $\chi_{\varepsilon,d} = W_\varepsilon - W_{\varepsilon,d} \in H_0^1(\Omega)$, we have

$$\text{div}(a_{\varepsilon,d}\nabla\chi_{\varepsilon,d}) = \text{div}((a_{\varepsilon,d} - a_\varepsilon)\nabla W_\varepsilon) = \text{div}((\gamma_1 - 1)\,\mathbf{1}_{G_\varepsilon}\,\nabla W_\varepsilon)$$

Thus, by testing against $\chi_{\varepsilon,d}$, we get

$$\|\nabla\chi_{\varepsilon,d}\|_{L^2(\Omega)} \leq C\|\nabla W_\varepsilon\|_{L^2(G_\varepsilon)} \leq C\,|G_\varepsilon|^{1/2}\,\|\nabla W_\varepsilon\|_{L^\infty(G_\varepsilon)}\,.$$

Next note that, thanks to Theorem 2, we have

$$\|\nabla W_\varepsilon\|_{L^\infty(G_\varepsilon)} \leq \|W_\varepsilon\|_{C^{1,\nu}(\Omega_\varepsilon)} \leq C\|\Phi_b\|_{C^1(\bar{\Omega})}, \tag{48}$$

with $\nu = 1/20$. Thus

$$\|\nabla\chi_{\varepsilon,d}\|_{L^2(\Omega)} \leq C\,|G_\varepsilon|^{1/2}\,\|\Phi_b\|_{C^1(\bar{\Omega})}\,. \tag{49}$$

This proves the first half of the desired estimate, namely

$$\|\nabla(W_\varepsilon - W_{\varepsilon,d})\|_{L^2(\Omega)} \le C \, |G_\varepsilon|^{1/2} \, \|\Phi_b\|_{C^1(\bar\Omega)}.$$

To prove the second part of the estimate, introduce the solution $F_\varepsilon \in H_0^1(\Omega)$ of

$$-\mathrm{div}\,(a_\varepsilon \nabla F_\varepsilon) = \chi_{\varepsilon,d} \ .$$

Thanks to Corollary 1, we have

$$\|\nabla F_\varepsilon\|_{L^6(\Omega_\varepsilon)} \le C\|\chi_{\varepsilon,d}\|_{L^2(\Omega)}.$$

On the other hand, note that

$$\mathrm{div}(a_\varepsilon \nabla\chi_{\varepsilon,d}) = \mathrm{div}((a_{\varepsilon,d} - a_\varepsilon)\nabla W_{\varepsilon,d}) = \mathrm{div}((\gamma_1 - 1)\,\mathbf{1}_{G_\varepsilon}\,\nabla W_{\varepsilon,d}).$$

After an integration by part, we see that

$$
\begin{aligned}
\|\chi_{\varepsilon,d}\|_{L^2(\Omega)}^2 &= \int_\Omega a_\varepsilon \nabla F_\varepsilon \cdot \nabla\chi_{\varepsilon,d}\,dx \\
&= \int_{G_\varepsilon} (\gamma_1 - 1)\,\mathbf{1}_{G_\varepsilon}\,\nabla F_\varepsilon \cdot \nabla W_{\varepsilon,d}\,dx \\
&= -\int_{G_\varepsilon} (\gamma_1 - 1)\,\nabla F_\varepsilon \cdot \nabla\chi_{\varepsilon,d}\,dx + \int_{G_\varepsilon} (\gamma_1 - 1)\,\nabla F_\varepsilon \cdot \nabla W_\varepsilon\,dx \\
&\le C\,\|\nabla F_\varepsilon\|_{L^6(\Omega_\varepsilon)}\left(\|\nabla\chi_{\varepsilon,d}\|_{L^{6/5}(G_\varepsilon)} + \|\nabla W_\varepsilon\|_{L^\infty(G_\varepsilon)}|G_\varepsilon|^{5/6}\right) \\
&\le C\|\chi_{\varepsilon,d}\|_{L^2(\Omega)}\left(\|\nabla\chi_{\varepsilon,d}\|_{L^2(\Omega)}|G_\varepsilon|^{5/6-1/2} + \|\nabla W_\varepsilon\|_{L^\infty(G_\varepsilon)}|G_\varepsilon|^{5/6}\right) \\
&\le C\|\chi_{\varepsilon,d}\|_{L^2(\Omega)}|G_\varepsilon|^{5/6}\|\Phi_b\|_{C^1(\bar\Omega)},
\end{aligned}
$$

where we have used (49) in the penultimate inequality. The proof is complete. $\quad\square$

The second ingredient is a pointwise uniform estimate on $W_\varepsilon - W_*$. We use the following extension of Theorem 3 with varying boundary data, the proof of which consists of a straightforward adaptation of the first step of the proof of Theorem 3.

Theorem 5 *Assume that the conditions (2), (3) and (12) hold. Consider a sequence $\varphi_\varepsilon \in C^1(\bar\Omega)$ which converges strongly in $C^0(\bar\Omega)$ to some $\varphi \in C^{0,1}(\bar\Omega)$. Then, the solution W_ε of problem (7) with the boundary condition $\Phi_b = \varphi_\varepsilon$ converges weakly in $H^1(\Omega)$ to the solution W_* of (13) with the boundary condition $\Phi_b = \varphi$. Furthermore, $W_* \in W_{\mathrm{loc}}^{2,p}(\Omega)$ and $V_* \in W_{\mathrm{loc}}^{2,p}(\Omega_0)$ for any $p > 2$.*

Proposition 6 *The solutions W_ε and W_* to respectively (6) and (13) satisfy the following convergence in the set Ω_ε of (45),*

$$\lim_{\varepsilon\to 0}\ \sup_{\substack{\Phi_b \in C^1(\bar\Omega) \\ \|\Phi_b\|_{C^1(\bar\Omega)} \le 1}}\ \|\nabla W_\varepsilon - \nabla W_*\|_{C^{0,1/20}(\Omega_\varepsilon)} = 0. \tag{50}$$

Proof By contradiction, suppose that there exist a constant $c_0 > 0$, a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$, and a sequence $\varphi_n \in C^1(\bar{\Omega})$ such that

$$\|\varphi_n\|_{C^1(\bar{\Omega})} \leq 1,$$

and

$$\|\nabla W_n - \nabla W_{*,n}\|_{C^{0,1/20}(\Omega_{\varepsilon_n})} \geq c_0,$$

where W_n is the solution of

$$\operatorname{div}(a_{\varepsilon_n} \nabla W_n) = 0 \text{ in } \Omega, \text{ and } W_n = \varphi_n \text{ on } \partial\Omega,$$

and $W_{*,n}$ is the solution of

$$\begin{cases} -\Delta W_{*,n} + \gamma \left(W_{*,n} - V_{*,n} \right) \mathbf{1}_{\Omega_0} = 0 \text{ in } \Omega \\ -\kappa\, \partial_{33}^2 V_{*,n} + \gamma \left(V_{*,n} - W_{*,n} \right) = 0 \text{ in } \Omega_0 = \omega_0 \times (-L, L) \\ \qquad\qquad\quad W_{*,n} = \varphi_n \text{ on } \partial\Omega \\ \quad V_{*,n}(\cdot, \pm L) = \varphi_n(\cdot, \pm L) \text{ in } \omega_0. \end{cases}$$

The regularity Theorem 2 shows that there exists a constant K, independent of n, such that

$$\|\nabla W_n\|_{C^{0,1/19}(\Omega_\varepsilon)} \leq K \|\varphi_n\|_{C^1(\bar{\Omega})} \leq K.$$

Then, by virtue of the extension theorem for Hölder functions (see, e.g., [30]) there exists an extension ξ_n of ∇W_n to the set $\Omega_1 = \omega_1 \times (-l, l)$, such that

$$\|\xi_n\|_{C^{0,1/19}(\Omega_1)^3} \leq K,$$

with the same constant K. Note that the embedding $C^{0,1/19}(\Omega_1) \hookrightarrow C^{0,1/20}(\Omega_1)$ is compact. Therefore we can extract a convergent subsequence ξ_p (and the associated sequence φ_p) such that

$$\|\xi_p - \xi\|_{C^{0,1/20}(\Omega_1)^3} \to 0.$$

Thanks to the Ascoli-Arzelà Theorem, we can extract yet another subsequence ξ_q, φ_q such that

$$\|\varphi_q - \varphi\|_{C^0(\bar{\Omega})} \to 0 \text{ for some } \varphi \in C^{0,1}(\bar{\Omega}).$$

Now, the homogenization Theorem 5 shows that

$$\nabla W_q \rightharpoonup \nabla W_* \text{ in } L^2(\Omega)^3,$$

where W_* is uniquely defined by

$$\begin{cases} -\Delta W_* + \gamma \left(W_* - V_* \right) \mathbf{1}_{\Omega_0} = 0 \text{ in } \Omega \\ -\kappa\, \partial_{33}^2 V_* + \gamma \left(V_* - W_* \right) = 0 \text{ in } \Omega_0 = \omega_0 \times (-L, L) \\ \qquad\qquad\quad W_* = \varphi \text{ on } \partial\Omega \\ \quad V_*(\cdot, \pm L) = \varphi(\cdot, \pm L) \text{ in } \omega_0. \end{cases}$$

A uniqueness argument shows that $\nabla W_* = \xi$ on Ω_1. In particular,

$$\|\nabla W_q - \nabla W_*\|_{C^{0,1/20}(\Omega_\varepsilon)} = o(1).$$

On the other hand, thanks to Theorem 3, the linearity of the homogenized system $W_{*,q} - W_* \in W^{2,4}(\Omega_1)$ and the Sobolev embedding Theorem we have,

$$\|\nabla W_{*,q} - \nabla W_*\|_{C^{0,1/20}(\Omega_\varepsilon)} \leq C \|\varphi_q - \varphi\|_{C^0(\Omega)} = o(1)$$

for a constant C independent of q, φ and ε. We therefore have obtained that

$$c_0 \leq \|\nabla W_q - \nabla W_{*,q}\|_{C^{0,1/20}(\Omega_\varepsilon)} = o(1),$$

which is a contradiction. $\square$

The third ingredient is an asymptotic representation formula for $\nabla (W_{\varepsilon,d} - W_\varepsilon)$ in the set G_ε,

Proposition 7 *(a) Let v_ε^i and $v_{\varepsilon,d}^i$ be solutions to (7) and (6), respectively for $\Phi_b^i = x_i$. Then, there exist a subsequence, still denoted by ε, and a matrix-valued function $M^* \in L^2(\Omega, \mu)^{3 \times 3}$, such that*

$$\lim_{\varepsilon \to 0} \int_\Omega \frac{\mathbf{1}_{G_\varepsilon}}{|G_\varepsilon|} \frac{\partial \left(v_{\varepsilon,d}^i - v_\varepsilon^i \right)}{\partial x_j} \psi \, dx = \int_\Omega M_{ij}^* \psi \, d\mu, \tag{51}$$

for all $\psi \in C^0(\Omega)$. Furthermore, the matrix M^* is symmetric, and satisfies

$$0 \leq M^*(y) \leq \frac{(\gamma_1 - 1)^2}{\gamma_1} \qquad \mu\text{-a.e.} \tag{52}$$

(b) We have, for all $w \in W^{2,4}(\Omega)$,

$$\int_\Omega (a_\varepsilon - a_{\varepsilon,d}) \nabla (W_{\varepsilon,d} - W_\varepsilon) \cdot \nabla w \, dx = |G_\varepsilon| \int_\Omega M^* \nabla W_* \cdot \nabla w \, d\mu + o(|G_\varepsilon|) \|w\|_{W^{2,4}(\Omega)},$$

where $o\left(|G_\varepsilon|\right) / |G_\varepsilon|$ converges to zero uniformly for $\|\Phi_b\|_{C^1(\bar{\Omega})} \leq 1$.

Remark 6 It is natural to ask if the extraction of a subsequence is necessary, even in the case of bounded conductivities. The answer is positive. In the case of domains G_ε shrinking to a point, there is an entire set of possible polarisation tensors for a given constant γ_1, delimited by the so-called Hashin-Strikman bounds, see e.g. [26, 29, 14, 2].

Remark 7 We can use a compactness argument to derive from (51) that

$$\lim_{\varepsilon \to 0} \sup_{\|\psi\|_{C^{0,\nu}(\Omega)} \leq 1} \left| \int_\Omega \frac{\mathbf{1}_{G_\varepsilon}}{|G_\varepsilon|} \frac{\partial \left(v_{\varepsilon,d}^i - v_\varepsilon^i \right)}{\partial x_j} \psi \, dx - \int_\Omega M_{ij}^* \psi \, d\mu \right| = 0$$

for any $0 < \nu \leq 1$. We remind the reader of Remark 4: μ has compact support in Ω. Indeed, if this fails for some ν, then there exist some $c_0 > 0$ and a sequence $\psi_n \in C^{0,\nu}(\Omega)$ such that (along some subsequence $\varepsilon \to 0$)

$$\left| \int_\Omega \frac{\mathbf{1}_{G_\varepsilon}}{|G_\varepsilon|} \frac{\partial \left(v_{\varepsilon,d}^i - v_\varepsilon^i \right)}{\partial x_j} \psi_n \, dx - \int_\Omega M_{ij}^* \psi_n \, d\mu \right| \geq c_0.$$

On the other hand, as the embedding $C^{0,\nu}(\Omega) \hookrightarrow C^0(\Omega)$ is compact, we can assume that $\psi_n \to \psi_*$ in $C^0(\Omega)$, which implies a violation to (51).

Proof The proof of this proposition is a variant of that of the main theorem of [13]. We provide the proof here for sake of completeness, adapted to the case at hand.

(a) Introducing v_ε^i and $v_{\varepsilon,d}^i$ solutions of solutions to (7) and (6), respectively for a given $\Phi_b^i = x_i$, we know thanks to Theorem 3 and Proposition 6 that

$$\lim_{\varepsilon \to 0} \left[\|v_\varepsilon^i - v_*^i\|_{L^2(\Omega)} + \|\nabla v_\varepsilon^i - \nabla v_*^i\|_{C^{0,1/20}(\Omega_\varepsilon)} \right] = 0,$$

where v_*^i is the solution of W_* of (13) corresponding to $\Phi_b = x_i$. It is immediate to verify that $v_*^i = x_i$. In particular, we have

$$\left\|v_\varepsilon^i - x_i\right\|_{W^{1,\infty}(\Omega_\varepsilon)} \le C \left(\left\|\nabla v_\varepsilon^i - e_i\right\|_{C^{0,1/20}(\Omega_\varepsilon)} + \left\|v_\varepsilon^i - x_i\right\|_{L^2(\Omega)} \right) = o(1), \tag{53}$$

Here and in the sequel, $o(1)$ denotes any quantity going to zero with ε, independently of Φ_b. On the other hand, using Lemma 2 we have

$$\left\|\nabla v_{\varepsilon,d}^i - \nabla v_\varepsilon^i\right\|_{L^2(\Omega)} \le C \, |G_\varepsilon|^{1/2} \ \text{ and } \ \left\|v_{\varepsilon,d}^i - v_\varepsilon^i\right\|_{L^2(\Omega)} \le C \, |G_\varepsilon|^{5/6}. \tag{54}$$

Applying the Cauchy-Schwarz inequality yields

$$\frac{1}{|G_\varepsilon|} \int_{G_\varepsilon} \left|\nabla v_{\varepsilon,d}^i - \nabla v_\varepsilon^i\right| \le \frac{1}{|G_\varepsilon|} \, |G_\varepsilon|^{1/2} \left\|\nabla v_{\varepsilon,d}^i - \nabla v_\varepsilon^i\right\|_{L^2(G_\varepsilon)} \le C, \tag{55}$$

thanks to (54). We may therefore extract a subsequence, still denoted by ε, such that

$$|G_\varepsilon|^{-1} \, \mathbf{1}_{G_\varepsilon} \, (1 - \gamma_1) \, \frac{\partial}{\partial x_j} \left(v_{\varepsilon,d}^i - v_\varepsilon^i\right) \, dx \longrightarrow d\mathcal{M}_{ij}, \tag{56}$$

where $\mathcal{M}_{i,j}$, $1 \le i, j \le 2$ is a Borel measure with support in $\bar\omega_0 \times [-l, l]$. The above convergence results hold in the weak-$*$ topology of $\mathcal{M}(\bar\omega_0 \times [-l, l])$ (the set of Radon measures on $\bar\omega_0 \times [-l, l]$). Note that $1 - \gamma_1 \in C^0(\Omega)$. We see that, for any $f \in C^0(\bar\Omega)$,

$$\left|\int_\Omega f \, d\mathcal{M}_{ij}\right| = \left|\lim_{\varepsilon \to 0} |G_\varepsilon|^{-1} \int_{G_\varepsilon} (1 - \gamma_1) \left(\frac{\partial}{\partial x_j} v_{\varepsilon,d}^i - \frac{\partial}{\partial x_j} v_\varepsilon^i\right) f \, dx\right|$$

$$\le \limsup_{\varepsilon \to 0} \frac{\|1 - \gamma_1\|_{L^\infty(\Omega)}}{\sqrt{|G_\varepsilon|}} \|\nabla v_{\varepsilon,d}^i - \nabla v_\varepsilon^i\|_{L^2(G_\varepsilon)} \left(\frac{1}{|G_\varepsilon|} \int_{G_\varepsilon} f^2 dx\right)^{1/2}.$$

Since $|G_\varepsilon|^{-1} \int_{G_\varepsilon} f^2 dx \to \int_{\Omega_0} f^2 d\mu$, we conclude, using (54) that

$$\left|\int_\Omega f \, d\mathcal{M}_{ij}\right| \le C \left(\int_{\Omega_0} |f|^2 \, d\mu\right)^{1/2}. \tag{57}$$

It follows that $d\mathcal{M}_{ij}$ is absolutely continuous with respect to μ and thus, there exists a 3×3 matrix valued function $M^* \in L^2(\Omega, \mu)^{3\times 3}$ such that

$$\int_{\Omega_0} f d\mathcal{M}_{ij} = \int_{\Omega_0} M_{ij}^* f d\mu.$$

Let us now turn to the properties of the matrix valued function M^*, following [13]. The matrix M^* is characterised by

$$\int_\Omega M_{ij}^* \, \psi \, d\mu = \frac{1}{|G_\varepsilon|} \int_{G_\varepsilon} (1 - \gamma_1) \, \nabla \left(v_{\varepsilon,d}^i - v_\varepsilon^i\right) \cdot \nabla x_j \, \psi \, dx + o(1)$$

for any $\psi \in C^0\left(\bar{\Omega}\right)$. Note that $v_{\varepsilon,d}^i - v_\varepsilon^i$ is the solution in $H_0^1(\Omega)$ of

$$\operatorname{div}\left(a_{\varepsilon,d}\nabla\left(v_{\varepsilon,d}^i - v_\varepsilon^i\right)\right) = \operatorname{div}\left((1-\gamma_1)\mathbf{1}_{G_\varepsilon}\nabla v_\varepsilon^i\right). \tag{58}$$

Using this identity together with Proposition 6 and Lemma 2, we obtain

$$\begin{aligned}
\int_\Omega M_{ij}^*\psi d\mu &= \frac{1}{|G_\varepsilon|}\int_{G_\varepsilon}(1-\gamma_1)\nabla\left(\left(v_{\varepsilon,d}^i - v_\varepsilon^i\right)\psi\right)\cdot\nabla v_\varepsilon^j dx + o(1) \\
&= \frac{1}{|G_\varepsilon|}\int_\Omega a_{\varepsilon,d}\nabla\left(\left(v_{\varepsilon,d}^i - v_\varepsilon^i\right)\psi\right)\cdot\nabla\left(v_{\varepsilon,d}^j - v_\varepsilon^j\right)dx + o(1) \\
&= \frac{1}{|G_\varepsilon|}\int_\Omega a_{\varepsilon,d}\nabla\left(v_{\varepsilon,d}^i - v_\varepsilon^i\right)\cdot\nabla\left(v_{\varepsilon,d}^j - v_\varepsilon^j\right)\psi \, dx + o(1).
\end{aligned}$$

Under this last form, it is apparent that $M_{ij}^* = M_{ji}^*$. Furthermore, given $\xi \in \mathbb{R}^3$, introducing $V_\varepsilon = \sum_{i=1}^3 (v_{\varepsilon,d}^i - v_\varepsilon^i)\xi_i$, this last identity yields

$$\int_\Omega M^*\xi\cdot\xi\psi d\mu = \int_\Omega \sum_{1\le i,j\le 3} M_{ij}^*\xi_i\,\xi_j\psi d\mu = \frac{1}{|G_\varepsilon|}\int_\Omega a_{\varepsilon,d}\left|\nabla V_\varepsilon\right|^2\psi \, dx + o(1). \tag{59}$$

This shows that $M^* \ge 0$, μ-almost everywhere. Alternatively, from (58) we derive, using Proposition 6 and Lemma 2, that for $\psi \ge 0$,

$$\begin{aligned}
\int_\Omega a_{\varepsilon,d}\left|\nabla V_\varepsilon\right|^2\psi dx &= \int_\Omega a_{\varepsilon,d}\nabla V_\varepsilon\cdot\nabla\left(V_\varepsilon\psi\right)dx + o(|G_\varepsilon|) \\
&= \int_\Omega\left((1-\gamma_1)\mathbf{1}_{G_\varepsilon}\right)\left(\sum_{i=1}^3\nabla v_\varepsilon^i\xi_i\right)\cdot\nabla\left(V_\varepsilon\psi\right)dx + o(|G_\varepsilon|) \\
&= \int_\Omega\left((1-\gamma_1)\mathbf{1}_{G_\varepsilon}\right)\xi\psi\cdot\nabla V_\varepsilon dx + o(|G_\varepsilon|) \\
&\le \left(\int_\Omega\frac{(1-\gamma_1)^2}{\gamma_1}\mathbf{1}_{G_\varepsilon}|\xi|^2\psi \, dx\right)^{1/2}\left(\int_\Omega a_{\varepsilon,d}\left|\nabla V_\varepsilon\right|^2\psi \, dx\right)^{1/2} + o(|G_\varepsilon|),
\end{aligned} \tag{60}$$

where we have used we have used (53) to derive (60). This shows that $M^* \le \frac{(1-\gamma_1)^2}{\gamma_1}$, μ-almost everywhere.

(b) Let us now show that

$$|G_\varepsilon|^{-1}\int_\Omega\left(a_\varepsilon - a_{\varepsilon,d}\right)\nabla\left(W_{\varepsilon,d} - W_\varepsilon\right)\cdot\nabla w dx = |G_\varepsilon|^{-1}\int_\Omega M^*\nabla W_*\cdot\nabla w d\mu + \operatorname{err}, \tag{61}$$

where err satisfies

$$|\operatorname{err}| = o(1)|G_\varepsilon|\,\|\Phi_b\|_{C^1(\bar{\Omega})}\|w\|_{W^{2,4}(\Omega)}.$$

Identity (61) can be interpreted as a 'separation of scales' result. On the left-hand side, the perturbation induced by the defect is present both in the 'microscopic' term $a_\varepsilon - a_{\varepsilon,d}$ and in the 'macroscopic' term $\nabla W_{\varepsilon,d} - \nabla W_\varepsilon$. On the right-hand side, the 'macroscopic' term ∇W_* is independent of the defect. To prove this result, we follow the adaptation of the oscillating test function method in homogenization of Murat & Tartar [31] proposed in [13]. In the following computation, $v_{\varepsilon,d}^i - v_\varepsilon^i$ plays the role of the corrector function in homogenization. By a chain of integration by parts, we will transfer the derivatives from $\nabla(W_{\varepsilon,d} - W_\varepsilon)$ to $\nabla(v_{\varepsilon,d}^i - v_\varepsilon^i)$, and then pass to the limit.

To express the left-hand side of (61) in terms of M^*, we introduce $v_{\varepsilon,d}^i - v_\varepsilon^i$ in the computation as follows. Using Einstein summation convention for the index i and noting that $v_*^i = x_i$, we have

$$\int_\Omega (a_\varepsilon - a_{\varepsilon,d}) \nabla (W_{\varepsilon,d} - W_\varepsilon) \cdot \nabla w\, dx = \int_{G_\varepsilon} (1 - \gamma_1) \nabla (W_{\varepsilon,d} - W_\varepsilon) \cdot \nabla v_*^i \frac{\partial w}{\partial x_i} dx \qquad (62)$$

$$= \int_{G_\varepsilon} (1 - \gamma_1) \nabla (W_{\varepsilon,d} - W_\varepsilon) \cdot \nabla v_\varepsilon^i \frac{\partial w}{\partial x_i} dx + \mathrm{err}_1,$$

with

$$|\mathrm{err}_1| \le C \|\nabla v_\varepsilon^i - \nabla v_*^i\|_{C^0(\Omega_\varepsilon)} \|\nabla w\|_{L^\infty(G_\varepsilon)} |G_\varepsilon|^{1/2} \|\nabla (W_{\varepsilon,d} - W_\varepsilon)\|_{L^2(G_\varepsilon)}$$
$$= o(1) |G_\varepsilon|\; \|\Phi_b\|_{C^1(\bar\Omega)} \|w\|_{W^{2,4}(\Omega)},$$

thanks to (54) and Lemma 2. Continuing the transformation, we write

$$\int_{G_\varepsilon} (1 - \gamma_1) \nabla (W_{\varepsilon,d} - W_\varepsilon) \cdot \nabla v_\varepsilon^i \frac{\partial w}{\partial x_i} dx = \int_{G_\varepsilon} (1 - \gamma_1) \nabla \left(\frac{\partial w}{\partial x_i} (W_{\varepsilon,d} - W_\varepsilon) \right) \cdot \nabla v_\varepsilon^i dx + \mathrm{err}_2 \quad (63)$$

$$= \int_\Omega a_{\varepsilon,d} \nabla \left(\frac{\partial w}{\partial x_i} (W_{\varepsilon,d} - W_\varepsilon) \right) \cdot \nabla \left(v_{\varepsilon,d}^i - v_\varepsilon^i \right) dx + \mathrm{err}_2,$$

with

$$|\mathrm{err}_2| \le C \|\nabla v_\varepsilon^i\|_{C^0(\Omega_\varepsilon)} \|\nabla^2 w\|_{L^2(G_\varepsilon)} \|(W_{\varepsilon,d} - W_\varepsilon)\|_{L^2(G_\varepsilon)} \le C |G_\varepsilon|^{\frac{13}{12}}\; \|\Phi_b\|_{C^1(\bar\Omega)} \|w\|_{W^{2,4}(\Omega)}$$
$$= o(1) |G_\varepsilon|\; \|\Phi_b\|_{C^1(\bar\Omega)} \|w\|_{W^{2,4}(\Omega)},$$

thanks to Lemma 2 and Hölder's inequality. To remove the $W_{\varepsilon,d}$ term, we write

$$\int_\Omega a_{\varepsilon,d} \nabla \left(\frac{\partial w}{\partial x_i} (W_{\varepsilon,d} - W_\varepsilon) \right) \cdot \nabla \left(v_{\varepsilon,d}^i - v_\varepsilon^i \right) dx \qquad (64)$$

$$= \int_\Omega a_{\varepsilon,d} \nabla (W_{\varepsilon,d} - W_\varepsilon) \cdot \nabla \left(\frac{\partial w}{\partial x_i} \left(v_{\varepsilon,d}^i - v_\varepsilon^i \right) \right) dx + \mathrm{err}_3$$

$$= \int_\Omega (a_\varepsilon - a_{\varepsilon,d}) \nabla W_\varepsilon \cdot \nabla \left(\frac{\partial w}{\partial x_i} \left(v_{\varepsilon,d}^i - v_\varepsilon^i \right) \right) dx + \mathrm{err}_3,$$

with

$$|\mathrm{err}_3| \le C \int_\Omega |\nabla^2 w| \left(|W_{\varepsilon,d} - W_\varepsilon| \left| \nabla \left(v_{\varepsilon,d}^i - v_\varepsilon^i \right) \right| + |\nabla (W_{\varepsilon,d} - W_\varepsilon)| \left| v_{\varepsilon,d}^i - v_\varepsilon^i \right| \right) dx$$
$$\le C \|w\|_{W^{2,4}(\Omega)} |G_\varepsilon|^{\frac{13}{12}}\; \|\Phi_b\|_{C^1(\bar\Omega)}$$
$$= o(1) |G_\varepsilon|\; \|\Phi_b\|_{C^1(\bar\Omega)} \|w\|_{W^{2,4}(\Omega)},$$

using Lemma 2, together with the interpolation inequality

$$\|f\|_{L^4(\Omega)} \le C \|f\|_{L^2(\Omega)}^{1/4} \|f\|_{L^6(\Omega)}^{3/4}.$$

Expanding this last expression, we have

$$\int_\Omega (a_\varepsilon - a_{\varepsilon,d}) \nabla W_\varepsilon \cdot \nabla \left(\frac{\partial w}{\partial x_i} \left(v_{\varepsilon,d}^i - v_\varepsilon^i \right) \right) dx \qquad (65)$$

$$= \int_{G_\varepsilon} (1 - \gamma_1) \nabla W_\varepsilon \cdot \nabla \left(v_{\varepsilon,d}^i - v_\varepsilon^i \right) \frac{\partial w}{\partial x_i} dx + \mathrm{err}_4,$$

with

$$|\text{err}_4| \leq C\|v_{\varepsilon,d}^i - v_\varepsilon^i\|_{L^2(G_\varepsilon)}\|\nabla W_\varepsilon\|_{L^\infty(G_\varepsilon)}|G_\varepsilon|^{1/2}\|w\|_{W^{1,\infty}(\Omega)}$$
$$= o(1)|G_\varepsilon|\,\|\Phi_b\|_{C^1(\bar\Omega)}\|w\|_{W^{2,4}(\Omega)}.$$

Using the convergence of W_ε to W_*, we get

$$\int_{G_\varepsilon} (1-\gamma_1)\nabla W_\varepsilon \cdot \nabla\left(v_{\varepsilon,d}^i - v_\varepsilon^i\right)\frac{\partial w}{\partial x_i}dx \tag{66}$$
$$= \int_\Omega \nabla W_* \cdot \left(\mathbf{1}_{G_\varepsilon}(1-\gamma_1)\nabla\left(v_{\varepsilon,d}^i - v_\varepsilon^i\right)\right)\frac{\partial w}{\partial x_i}dx + \text{err}_5,$$

with

$$|\text{err}_5| \leq C\|\left(\mathbf{1}_{G_\varepsilon}\nabla\left(v_{\varepsilon,d}^i - v_\varepsilon^i\right)\right)\frac{\partial w}{\partial x_i}\|_{L^1(G_\varepsilon)}\|\nabla W_\varepsilon - \nabla W_*\|_{L^\infty(G_\varepsilon)}$$
$$\leq C\,|G_\varepsilon|\,\|w\|_{W^{2,4}(\Omega)}\|\nabla W_\varepsilon - \nabla W_*\|_{L^\infty(G_\varepsilon)}.$$

Note that thanks to Proposition 6, we know that

$$\|\nabla W_\varepsilon - \nabla W_*\|_{L^\infty(G_\varepsilon)} = o(1)\|\Phi_b\|_{C^1(\bar\Omega)},$$

therefore $|\text{err}_5| = o(1)|G_\varepsilon|\,\|\Phi_b\|_{C^1(\bar\Omega)}\|w\|_{W^{2,4}(\Omega)}$. Finally, by Remark 7,

$$\frac{1}{|G_\varepsilon|}\int_\Omega \nabla W_* \cdot \left(\mathbf{1}_{G_\varepsilon}(1-\gamma_1)\nabla\left(v_{\varepsilon,d}^i - v_\varepsilon^i\right)\right)\frac{\partial w}{\partial x_i}dx = \int_\Omega \nabla W_* \cdot (M^*)^T\nabla w\,d\mu + \text{err}_6 \tag{67}$$

where

$$|\text{err}_6| = o(1)\|\nabla W_*\|_{C^{0,1/20}(\Omega)}\|\nabla w\|_{C^{0,1/20}(\Omega_\varepsilon)} = o(1)\,\|\Phi_b\|_{C^1(\bar\Omega)}\|w\|_{W^{2,4}(\Omega)}.$$

Combining (62),(63),(64),(65),(66) and (67), we obtain (61). $\square$

Proof of Theorem 4. A straight-forward integration by parts shows that

$$\mathscr{R}_\varepsilon(\Phi_b) = \int_{G_\varepsilon}(\gamma_1 - 1)\nabla W_{\varepsilon,d} \cdot \nabla W_\varepsilon\,dx, \tag{68}$$

which we rewrite in the form

$$\mathscr{R}_\varepsilon(\Phi_b) = \int_{G_\varepsilon}(\gamma_1 - 1)\nabla W_* \cdot \nabla W_*\,dx$$
$$+ \int_{G_\varepsilon}(a_{\varepsilon,d} - a_\varepsilon)\nabla(W_{\varepsilon,d} - W_\varepsilon) \cdot \nabla W_*\,dx + r_\varepsilon,$$
$$\text{with}\quad r_\varepsilon = \int_{G_\varepsilon}(\gamma_1 - 1)(\nabla W_\varepsilon \cdot \nabla W_\varepsilon - \nabla W_* \cdot \nabla W_*)\,dx$$
$$+ \int_{G_\varepsilon}(a_{\varepsilon,d} - a_\varepsilon)\nabla(W_{\varepsilon,d} - W_\varepsilon) \cdot \nabla(W_\varepsilon - W_*)\,dx.$$

By Proposition 6, Lemma 2, Theorem 3 and (48), we have

$$|r_\varepsilon| \leq C\,|G_\varepsilon|\,\|\nabla W_\varepsilon - \nabla W_*\|_{C^0(\Omega_\varepsilon)^3}\,\|\Phi_b\|_{C^1(\bar\Omega)} = o(|G_\varepsilon|).$$

On the other hand, Proposition 7 shows that

$$\int_{G_\varepsilon} (a_{\varepsilon,d} - a_\varepsilon) \nabla (W_{\varepsilon,d} - W_\varepsilon) \cdot \nabla W_* \, dx = -|G_\varepsilon| \int_\Omega M^* \nabla W_* \cdot \nabla W_* \, d\mu + o(|G_\varepsilon|) \|W_*\|_{W^{2,4}(\Omega)},$$

and this establishes the representation formula given by Theorem 4, thanks to Theorem 3 with

$$M_{ij} = (\gamma_1 - 1)\delta_{ij} - M_{ij}^*.$$

The bounds (52) on M^* imply the announced bounds on M. $\square$

To conclude this section, we now provide an alternative characterisation of M^*, following [15].

Proposition 8 *Let M^* be the polarisation tensor introduced by Proposition 7. Let ψ be a uniformly positive, smooth function on Ω, and $\xi \in \mathbb{R}^3$. Then M^* satisfies*

$$\int_\Omega M^* \xi \cdot \xi \, \psi \, d\mu = \int_\Omega \frac{(\gamma_1 - 1)^2}{\gamma_1} |\xi|^2 \, \psi \, d\mu$$
$$- \frac{1}{|G_\varepsilon|} \min_{w \in H_0^1(\Omega)} \int_\Omega a_{\varepsilon,d} \left|\nabla w + \frac{\gamma_1 - 1}{\gamma_1} \mathbf{1}_{G_\varepsilon} \xi\right|^2 \psi \, dx + o(1),$$

where $o(1)$ may depend on ψ but goes to zero with ε.

Proof Let ζ_ε be the solution in $H_0^1(\Omega)$ of

$$\mathrm{div}\,(a_{\varepsilon,d}\psi\nabla\zeta_\varepsilon) = \mathrm{div}\,(\psi\,(1 - \gamma_1)\,\mathbf{1}_{G_\varepsilon}\xi). \tag{69}$$

Note that ζ_ε is the unique minimizer of

$$\min_{w \in H_0^1(\Omega)} \int_\Omega a_{\varepsilon,d} \left|\nabla w + \frac{\gamma_1 - 1}{\gamma_1} \mathbf{1}_{G_\varepsilon} \xi\right|^2 \psi \, dx.$$

In particular,

$$\int_\Omega a_{\varepsilon,d} \left(\nabla\zeta_\varepsilon + \frac{\gamma_1 - 1}{\gamma_1}\mathbf{1}_{G_\varepsilon}\xi\right) \cdot \nabla\zeta_\varepsilon \psi \, dx = 0,$$

and therefore, as $\gamma_1 \in C^0(\Omega)$,

$$\int_\Omega a_{\varepsilon,d} \left|\nabla\zeta_\varepsilon + \frac{\gamma_1 - 1}{\gamma_1}\mathbf{1}_{G_\varepsilon}\xi\right|^2 \psi \, dx$$
$$= -\int_\Omega a_{\varepsilon,d} |\nabla\zeta_\varepsilon|^2 \psi \, dx + \int_{G_\varepsilon} \frac{(\gamma_1 - 1)^2}{\gamma_1} |\xi|^2 \psi \, dx$$
$$- \int_\Omega a_{\varepsilon,d} |\nabla\zeta_\varepsilon|^2 \psi \, dx + \int_\Omega \frac{(\gamma_1 - 1)^2}{\gamma_1} |\xi|^2 \psi \, d\mu + o(1). \tag{70}$$

Let us prove that ζ_ε satisfies an estimate similar to that of Lemma 2, namely

$$\|\nabla\zeta_\varepsilon\|_{L^2(\Omega)^3} + |G_\varepsilon|^{-1/3} \|\zeta_\varepsilon\|_{L^2(G_\varepsilon)} \le C |G_\varepsilon|^{1/2}, \tag{71}$$

where C may depend on ψ, but is independent of ε. Testing (69) against ζ_ε, and integrating by parts, we obtain

$$\|\nabla \zeta_\varepsilon\|^2_{L^2(\Omega)^3} \leq \frac{1}{\min \psi} \int_\Omega a_{\varepsilon,d} \psi \, |\nabla \zeta_\varepsilon|^2 \, dx$$

$$= \frac{1}{\min \psi} \int_{G_\varepsilon} (1 - \gamma_1) \, \xi \cdot \nabla \zeta_\varepsilon$$

$$\leq C \, |G_\varepsilon|^{1/2} \, \|\nabla \zeta_\varepsilon\|_{L^2(\Omega)^3} \, ,$$

using the Cauchy-Schwarz inequality. On the other hand, using Hölder's inequality followed by Poincaré-Sobolev's inequality,

$$\|\zeta_\varepsilon\|_{L^2(G_\varepsilon)} \leq |G_\varepsilon|^{1/3} \, \|\zeta_\varepsilon\|_{L^6(\Omega)} \leq C \, |G_\varepsilon|^{1/3} \, \|\nabla \zeta_\varepsilon\|_{L^2(\Omega)},$$

and (71) is established. Introducing $V_\varepsilon = \sum_{i=1}^3 (v_{\varepsilon,d}^i - v_\varepsilon^i)\xi_i$, the identity (60) shows that

$$\int_\Omega a_{\varepsilon,d} \, |\nabla V_\varepsilon|^2 \, \psi dx = \int_\Omega ((1 - \gamma_1)\mathbf{1}_{G_\varepsilon}) \, \xi \psi \cdot \nabla V_\varepsilon dx + o(|G_\varepsilon|)$$

$$= \int_\Omega a_{\varepsilon,d} \psi \nabla \zeta_\varepsilon \cdot \nabla V_\varepsilon dx + o(|G_\varepsilon|).$$

On the other hand, using Proposition 6, followed by (71), used twice,

$$\int_\Omega a_{\varepsilon,d} \, |\nabla \zeta_\varepsilon|^2 \, \psi dx = \int_\Omega ((1 - \gamma_1)\mathbf{1}_{G_\varepsilon}) \, \xi \psi \cdot \nabla \zeta_\varepsilon dx$$

$$= \int_\Omega ((1 - \gamma_1)\mathbf{1}_{G_\varepsilon}) \left(\sum_{i=1}^3 \nabla v_\varepsilon^i \xi_i \right) \cdot \nabla \zeta_\varepsilon \psi dx + o(|G_\varepsilon|)$$

$$= \int_\Omega ((1 - \gamma_1)\mathbf{1}_{G_\varepsilon}) \left(\sum_{i=1}^3 \nabla v_\varepsilon^i \xi_i \right) \cdot \nabla (\zeta_\varepsilon \psi) \, dx + o(|G_\varepsilon|)$$

$$= \int_\Omega a_{\varepsilon,d} \psi \nabla V_\varepsilon \cdot \nabla \zeta_\varepsilon dx + o(|G_\varepsilon|).$$

We have obtained that

$$\int_\Omega a_{\varepsilon,d} \, |\nabla V_\varepsilon|^2 \, \psi dx = \int_\Omega a_{\varepsilon,d} \, |\nabla \zeta_\varepsilon|^2 \, \psi dx + o(|G_\varepsilon|).$$

The conclusion then follows directly from the above identity, (59) and (70). $\quad\square$

5 Proof of the homogenization result

This section provides a proof of Theorem 3. The proof is divided in three steps.

First step: Derivation of the homogenization problem (13).
This step uses the same ingredients of [12] and [5], but the boundary conditions are different. Choosing $W_\varepsilon - \Phi_b \in H_0^1(\Omega)$ as a test function in equation (7), we obtain

$$\int_\Omega a_\varepsilon |\nabla W_\varepsilon|^2 \, dx = \int_\Omega a_\varepsilon \nabla W_\varepsilon \cdot \nabla \Phi_b \, dx = \int_{Q_\varepsilon} \alpha_\varepsilon \nabla W_\varepsilon \cdot \nabla \Phi_b \, dx + \int_{\Omega \backslash Q_\varepsilon} \nabla W_\varepsilon \cdot \nabla \Phi_b \, dx.$$

This combined with (3), (12) and the Cauchy-Schwarz inequality yields

$$\int_\Omega a_\varepsilon |\nabla W_\varepsilon|^2 \, dx \le c \, \|\nabla \Phi_b\|_{L^\infty(\Omega)} \left(\int_\Omega a_\varepsilon |\nabla W_\varepsilon|^2 \, dx \right)^{1/2},$$

hence the energy estimate

$$\int_\Omega a_\varepsilon |\nabla W_\varepsilon|^2 \, dx \le c \, \|\nabla \Phi_b\|_{L^\infty(\Omega)}^2. \tag{72}$$

Estimate (72) implies that ∇W_ε is bounded in $L^2(\Omega)$. Due to the Dirichlet boundary condition and the regularity of Ω, the sequence W_ε is bounded in $H^1(\Omega)$, and thus converges weakly to some function W_* in $H^1(\Omega)$ up to a subsequence.

Integrating along vertical lines, we can prove (see [5], [12] for further details) that the rescaled function $V_\varepsilon = (\pi r_\varepsilon^2)^{-1} \, 1_{Q_\varepsilon} \, W_\varepsilon$ converges weakly-$*$ in $\mathcal{M}(\Omega_0)$ (i.e., in the sense of measures on Ω_0) to some function $V_* \in H^1\big((-L, L); L^2(\omega_0)\big)$. Moreover, the uniform repartition of the highly conductive cylinders $Q_{m,n,\varepsilon}$ and the continuity of Φ_b imply that V_* inherits of the Dirichlet boundary condition on $\omega_0 \times \{\pm L\}$.

On the one hand, by (12) and (72) we have

$$a_\varepsilon \, \partial_3 W_\varepsilon \, 1_{Q_\varepsilon} = \partial_3 \left(\pi r_\varepsilon^2 \, \alpha_\varepsilon \, V_\varepsilon \right) \; \rightharpoonup \; \kappa \, \partial_3 V_* \quad \text{weakly-} * \text{ in } \mathcal{M}(\Omega_0),$$

and there is no transverse diffusion induced by the cylinders $Q_{m,n,\varepsilon}$ (see [12]). Moreover, due to the energy estimate (72) and the periodicity of a_ε, there is no concentration effect of $a_\varepsilon \nabla W_\varepsilon$ on $\partial \Omega_0$. Therefore, we obtain the convergences of the flux

$$\begin{cases} a_\varepsilon \nabla W_\varepsilon \, c_\varepsilon^\sigma & \rightharpoonup \nabla W_* \qquad \text{weakly-} * \text{ in } \mathcal{M}(\bar{\Omega}) \\ a_\varepsilon \nabla W_\varepsilon \, (1 - c_\varepsilon^\sigma) & \rightharpoonup \kappa \, \partial_3 V_* \, e_3 \; \text{weakly-} * \text{ in } \mathcal{M}(\bar{\Omega}_0). \end{cases} \tag{73}$$

On the other hand, the sequence c_ε^σ defined by (11) satisfies the convergences (see [12])

$$c_\varepsilon^\sigma \rightharpoonup 1 \;\; \text{weakly in } H^1(\Omega) \quad \text{and} \quad |\nabla c_\varepsilon^\sigma|^2 \rightharpoonup \gamma \;\; \text{weakly-} * \text{ in } \mathcal{M}(\bar{\Omega}_0). \tag{74}$$

Then, thanks to [12, Lemma 2] combined with (3), and again using the fact that there is no concentration effect on $\partial \Omega_0$, we get the convergence

$$\nabla W_\varepsilon \cdot \nabla c_\varepsilon^\sigma \; \rightharpoonup \; \gamma \, (W_* - V_*) \quad \text{weakly-} * \text{ in } \mathcal{M}(\bar{\Omega}_0), \tag{75}$$

which induces the non-local effect in the homogenization process.

Let $\psi_0, \psi_1 \in C_c^1(\Omega)$. Choosing $\psi_0 \, c_\varepsilon^\sigma + \psi_1 \, (1 - c_\varepsilon^\sigma)$ as a test function in equation (13), and passing to the limit as $\varepsilon \to 0$ thanks to the convergences (73), (74), (75), we obtain the equality

$$\int_\Omega \nabla W_* \cdot \nabla \psi_0 \, dx + \int_{\Omega_0} \kappa \, \partial_3 V_* \, \partial_3 \psi_1 \, dx + \int_{\Omega_0} \gamma \, (W_* - V_*) \, (\psi_0 - \psi_1) \, dx = 0,$$

which corresponds to the weak formulation of problem (13).

Second step: Proof of the local regularity of W_* and V_*.

Rewrite the first and the second equations of problem (13) as

$$- \Delta W_* = -\gamma \, 1_{\Omega_0} \, (W_* - V_*) \quad \text{in } \Omega, \tag{76}$$

and

$$- \kappa \, \partial_{33}^2 V_* + \gamma \, V_* = \gamma \, W_* \quad \text{in } \Omega_0 = \omega_0 \times (-L, L). \tag{77}$$

As $W_* \in H^1(\Omega)$ and $\Phi_b \in C^1(\bar{\Omega})$, $V_* \in H^1(\Omega_0)$. By the Sobolev embedding theorem, the right-hand side of (76) is in $L^6(\Omega)$. Thus, as $\partial\Omega$ is Lipschitz thus satisfies the exterior cone condition, the de Giorgi - Nash - Moser estimate (see e.g. [20, Theorems 8.22, 8.27]), $W_* \in C^{0,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$. Consequently, $V_*, \partial_{x_3}V_*, \partial_{x_3}^2 V_* \in C^{0,\alpha}(\bar{\Omega}_0)$.

Going back to (76), the right-hand side belongs to $L^\infty(\Omega)$. Thus, $W_* \in W_{\text{loc}}^{2,p}(\Omega)$ for any $p > 2$. Bootstrapping between (76) and (77) we obtain $W_* \in C_{\text{loc}}^\infty(\Omega_0) \cap C_{\text{loc}}^\infty(\Omega \setminus \Omega_0)$ and $V_* \in C_{\text{loc}}^\infty(\Omega_0)$.

Third step: Proof of the corrector result (14).

Denote by E_ε the left-hand side of (14). By equation (7) we have

$$\int_\Omega a_\varepsilon |\nabla W_\varepsilon|^2 \, dx = \int_\Omega a_\varepsilon \nabla W_\varepsilon \cdot \nabla \Phi_b \, dx.$$

Hence, by the definition (11) of c_ε^σ, it follows that

$$\begin{aligned}
E_\varepsilon = {} & \int_\Omega a_\varepsilon \nabla W_\varepsilon \cdot \nabla \Phi_b \, dx + \int_{\Omega_0} |\nabla c_\varepsilon^\sigma|^2 \, (W_* - V_*)^2 \, dx \\
& + \int_\Omega (c_\varepsilon^\sigma)^2 \, |\nabla W_*|^2 \, dx + \int_{\Omega_0} a_\varepsilon (1 - c_\varepsilon^\sigma)^2 \, (\partial_3 V_*)^2 \, dx \\
& - 2 \int_{\Omega_0} \nabla W_\varepsilon \cdot \nabla c_\varepsilon^\sigma \, (W_* - V_*) \, dx - 2 \int_\Omega c_\varepsilon^\sigma \, \nabla W_\varepsilon \cdot \nabla W_* \, dx \\
& - 2 \int_{\Omega_0} a_\varepsilon (1 - c_\varepsilon^\sigma) \, \partial_3 W_\varepsilon \, \partial_3 V_* \, dx + o(1).
\end{aligned}$$

Then, passing to the limit as $\varepsilon \to 0$ thanks to the convergences (73), (74), (75) combined with the continuity of the functions $W_* - V_*$ and $\partial_3 V_*$, we obtain

$$\begin{aligned}
\limsup_{\varepsilon \to 0} E_\varepsilon \leq E_0 = {} & \int_\Omega \nabla W_* \cdot \nabla \Phi_b \, dx + \int_{\Omega_0} \kappa \, \partial_3 V_* \, \partial_3 \Phi_b \, dx \\
& + \int_{\Omega_0} \gamma \, (W_* - V_*)^2 \, dx + \int_\Omega |\nabla W_*|^2 \, dx + \int_{\Omega_0} \kappa \, (\partial_3 V_*)^2 \, dx \\
& - 2 \int_{\Omega_0} \gamma \, (W_* - V_*)^2 \, dx - 2 \int_\Omega |\nabla W_*|^2 \, dx - 2 \int_{\Omega_0} \kappa \, (\partial_3 V_*)^2 \, dx \\
= {} & \int_\Omega \nabla W_* \cdot \nabla \Phi_b \, dx + \int_{\Omega_0} \kappa \, \partial_3 V_* \, \partial_3 \Phi_b \, dx - \int_{\Omega_0} \gamma \, (W_* - V_*)^2 \, dx \\
& - \int_\Omega |\nabla W_*|^2 \, dx - \int_{\Omega_0} \kappa \, (\partial_3 V_*)^2 \, dx.
\end{aligned}$$

Here we have used

$$a_\varepsilon (1 - c_\varepsilon^\sigma)^2 \rightharpoonup \lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon^2} \int_{\varepsilon Y} a_\varepsilon (1 - c_\varepsilon^\sigma)^2 \, dx \right) = \lim_{\varepsilon \to 0} \left(\alpha_\varepsilon \, \pi r_\varepsilon^2 \right) = \kappa \quad \text{weakly-$*$ in } \mathcal{M}(\bar{\Omega}_0).$$

On the other hand, choosing $W_* - \Phi_b$ as test function in the first equation of (13) and $V_* - \Phi_b$ in the second equation of (13), it follows that

$$\begin{aligned}
\int_\Omega \nabla W_* \cdot \nabla (W_* - \Phi_b) \, dx + \int_\Omega \gamma \, (W_* - V_*) \, (W_* - \Phi_b) \, dx = 0 \\
\int_{\Omega_0} \kappa \, \partial_3 V_* \, \partial_3 (V_* - \Phi_b) \, dx - \int_{\Omega_0} \gamma \, (W_* - V_*) \, (V_* - \Phi_b) \, dx = 0.
\end{aligned}$$

Therefore, adding the two previous equalities we obtain that $E_0 = 0$, which gives the thesis.

6 On supremum estimates

A less *ad hoc* version of Lemma 1 is given below. It is probably known to the experts.

Lemma 3 *For $\Omega \subset \mathbb{R}^n$, assume that $w \in H_0^1(\Omega)$ satisfies*

$$-\partial_i(A^{ij}(x)\partial_j w) + c(x)\, w \leq f + \operatorname{div}(\mathbf{h}) \text{ in } \Omega \, ,$$

for some $f \in L^p(\Omega)$, $\mathbf{h} \in L^{2\tilde{p}}(\Omega)^n$, $p, \tilde{p} > \frac{n}{2}$. If the coefficients A^{ij} and c are locally bounded and satisfy

$$A^{ij}(x)\, \xi_i\, \xi_j \geq \nu\, |\xi|^2 \text{ for any } x \in \Omega \text{ and } \xi \in \mathbb{R}^n \, ,$$
$$c(x) \geq \lambda \text{ for any } x \in \Omega \, ,$$

then $w^+ \in L^\infty(\Omega)$ and, for any $q > \frac{2p}{2p-n}$ and $\tilde{q} > \frac{2\tilde{p}}{2\tilde{p}-n}$,

$$\|w^+\|_{L^\infty(\Omega)} \leq C(n, q, \nu) \left[\frac{1}{\lambda^{\frac{1}{q}}} \|f\|_{L^p(\Omega)} + \frac{1}{\lambda^{\frac{1}{2\tilde{q}}}} \|\mathbf{h}\|_{L^{2p}(\Omega)} \right] \, .$$

Proof of Lemma 1. We follow de Giorgi's method for the proof. For the sake of Lemma 3 we will assume that

$$\int_\omega a_\varepsilon f^p dx < \infty \text{ and } \int_\omega a_\varepsilon \mathbf{h}^{2\tilde{p}} dx < \infty$$

for some $p, \tilde{p} > 1$; note that $n = 2$ in the present proof.

For $k > 0$, let $A(k) = \{x \in \omega : \varphi_\varepsilon(x) \geq k\}$. Let B represent the quantity

$$B = \int_{A(k)} a_\varepsilon \left| \nabla\left(\varphi_\varepsilon - k\right)_+ \right|^2 + \lambda \int_{A(k)} a_\varepsilon \left(\varphi_\varepsilon - k\right)_+^2 + \lambda \int_{A(k)} a_\varepsilon k \left(\varphi_\varepsilon - k\right)_+ \, .$$

Integrating (22) by parts against $(u - k)_+$, we obtain

$$B = \int_{A(k)} a_\varepsilon f(u - k)_+ - \int_{A(k)} a_\varepsilon \mathbf{h} \cdot \nabla(u - k)_+. \tag{78}$$

We shall use the notation conventions that

$$\|f\|_{\mathcal{L}^p(A_k)} = \left(\int_{A_k} a_\varepsilon f^p dx \right)^{1/p} \text{ and } N(A_k) = \int_{A_k} a_\varepsilon dx.$$

Using the weighted Sobolev embedding given by Lemma 4 we obtain

$$\mu_\varepsilon \left\|(\varphi_\varepsilon - k)_+\right\|_{\mathcal{L}^s(A_k)}^2 + \lambda \left\|(\varphi_\varepsilon - k)_+\right\|_{\mathcal{L}^2(A_k)}^2 \leq B,$$

where

$$\mu_\varepsilon := C(s)\, \varepsilon^{2(1-2/s)}. \tag{79}$$

Using Young's inequality, we have, for any $1 > \theta_1 > 0$ and $\theta_2 = 1 - \theta_1$,

$$\mu_\varepsilon \left\|(\varphi_\varepsilon - k)_+\right\|_{\mathcal{L}^s(A_k)}^2 + \lambda \left\|(\varphi_\varepsilon - k)_+\right\|_{\mathcal{L}^2(A_k)}^2$$

$$\geq \left(\frac{\mu_\varepsilon}{\theta_1}\right)^{\theta_1} \left(\frac{\lambda}{\theta_2}\right)^{\theta_2} \left\|(\varphi_\varepsilon - k)_+\right\|_{\mathcal{L}^s(A_k)}^{2\theta_1} \left\|(\varphi_\varepsilon - k)_+\right\|_{\mathcal{L}^2(A_k)}^{2\theta_2}$$

$$= \left(\frac{\mu_\varepsilon}{\theta_1}\right)^{\theta_1} \left(\frac{\lambda}{\theta_2}\right)^{\theta_2} \left\|(\varphi_\varepsilon - k)_+^{r\theta_1}\right\|_{\mathcal{L}^{s/r\theta_1}(A_k)}^{2/r} \left\|(\varphi_\varepsilon - k)_+^{r\theta_2}\right\|_{\mathcal{L}^{2/r\theta_2}(A_k)}^{2/r} \, .$$

On the other hand, Holder's inequality shows that

$$\left\|\left(\varphi_\varepsilon - k\right)_+\right\|^2_{\mathcal{L}^r(A_k)} = \left\|\left(\varphi_\varepsilon - k\right)^{\theta_1 r}_+ \left(\varphi_\varepsilon - k\right)^{\theta_2 r}_+\right\|^{2/r}_{\mathcal{L}^1(A_k)}$$

$$\leq \left\|\left(\varphi_\varepsilon - k\right)^{\theta_1 r}_+\right\|^{2/r}_{\mathcal{L}^\alpha(A_k)} \left\|\left(\varphi_\varepsilon - k\right)^{\theta_2 r}_+\right\|^{2/r}_{\mathcal{L}^\beta(A_k)},$$

for any $1 < \alpha, \beta < \infty$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Choosing $\alpha = \frac{s}{r\theta_1}$ yields $\beta = \frac{2}{r\theta_2}$ provided

$$\frac{1}{r} = \frac{\theta_1}{s} + \frac{\theta_2}{2}.$$

We have obtained

$$\left(\frac{\mu_\varepsilon}{\theta_1}\right)^{\theta_1} \left(\frac{\lambda}{\theta_2}\right)^{\theta_2} \left\|\left(\varphi_\varepsilon - k\right)_+\right\|^2_{\mathcal{L}^r(A_k)} \tag{80}$$

$$\leq \left\|\nabla\left(\varphi_\varepsilon - k\right)_+\right\|^2_{\mathcal{L}^2(A_k)} + \lambda \left\|\left(\varphi_\varepsilon - k\right)_+\right\|^2_{\mathcal{L}^2(A_k)} \leq B.$$

Let us now turn to the right-hand side.

Case 1: $\mathbf{h} = 0$. We have

$$B \leq \left\|f\left(\varphi_\varepsilon - k\right)_+\right\|_{\mathcal{L}^1(A_k)}$$

$$\leq \|f\|_{\mathcal{L}^p(A_k)} \left\|\left(\varphi_\varepsilon - k\right)_+\right\|_{\mathcal{L}^r(A_k)} \|1\|_{\mathcal{L}^\kappa(A_k)}$$

with $\frac{1}{\kappa} + \frac{1}{p} + \frac{1}{r} = 1$. We require that θ_1, θ_2 be chosen so that $\kappa > 1$. For $p = 2$ and $s > 2$, this requirement is fulfilled by any $\theta_1 > 0$ since $r \in (2, s)$. For the general case when $p > \frac{n}{2}$, we can satisfy the above requirement by selecting s sufficiently large (but smaller than the Sobolev exponent) so that $\frac{1}{p} + \frac{1}{s} < 1$. We have obtained

$$\left\|\left(\varphi_\varepsilon - k\right)_+\right\|_{\mathcal{L}^r(A_k)} \leq \left(\frac{\theta_1}{\mu_\varepsilon}\right)^{\theta_1} \left(\frac{\theta_2}{\lambda}\right)^{\theta_2} \|f\|_{\mathcal{L}^p(\omega)} N\left(A_k\right)^{\chi/r}, \tag{81}$$

where $\chi = r\left(1 - p^{-1} - r^{-1}\right) = r/\kappa$. Now, for all $h < k$,

$$N\left(A_k\right) = \int_{A_k} a_\varepsilon \, dx \leq \int_{A_k} a_\varepsilon \frac{(w - h)^r_+}{(k - h)^r} \, dx = \left\|\left(\varphi_\varepsilon - h\right)_+\right\|^r_{\mathcal{L}^r(A_h)} \frac{1}{|k - h|^r}.$$

Therefore

$$\left\|\left(\varphi_\varepsilon - k\right)_+\right\|_{\mathcal{L}^r(A_k)} \leq \left(\frac{\theta_1}{\mu_\varepsilon}\right)^{\theta_1} \left(\frac{\theta_2}{\lambda}\right)^{\theta_2} \frac{1}{|k - h|^\chi} \|f\|_{\mathcal{L}^p(\omega)} \left\|\left(\varphi_\varepsilon - h\right)_+\right\|^\chi_{\mathcal{L}^r(A_h)}.$$

To proceed, we select θ_1 such that $\theta_1 > \frac{s}{p(s-2)}$. Then $\chi > 1$.

We now set $k_j = 2d\left(1 - 2^{-j-1}\right)$. Introducing

$$\beta = \|f\|_{\mathcal{L}^p(\omega)} \left(\frac{\theta_1}{\mu_\varepsilon}\right)^{\theta_1} \left(\frac{\theta_2}{\lambda}\right)^{\theta_2} 2^\chi d^{-\chi}, \text{ and } x_j = \beta^{\frac{1}{\chi-1}} \left\|\left(\varphi_\varepsilon - k_j\right)_+\right\|_{\mathcal{L}^r\left(A_{k_j}\right)},$$

Then (81) takes the form

$$x_{j+1} \leq 2^{j\chi} x_j^\chi.$$

An induction shows that
$$\text{if } x_0 \leq 2^{-\chi/(\chi-1)^2}, \text{ then } x_j \leq 2^{-\frac{j\chi}{\chi-1}} x_0.$$

Therefore, $\lim_{j\to\infty} x_j = 0$, and $\left\|(\varphi_\varepsilon)_+\right\|_{L^\infty(\omega)} \leq 2d$. Thus, d is given by the constraint
$$x_0 \leq 2^{-\frac{\chi}{(\chi-1)^2}}.$$

Using the L^∞ bound just derived, for $j = 0$, we obtain
$$x_0 = \beta^{\frac{1}{\chi-1}} \left\|(\varphi_\varepsilon - d)_+\right\|_{\mathcal{L}^r(A_0)} \leq N(A_0)^{\frac{1}{r}} \beta^{\frac{1}{\chi-1}} d.$$

Therefore we choose d to be
$$d = \|f\|_{\mathcal{L}^p(\omega)} \left(\frac{\theta_1}{\mu_\varepsilon}\right)^{\theta_1} \left(\frac{\theta_2}{\lambda}\right)^{\theta_2} N(A_0)^{\frac{\chi-1}{r}} 2^{\frac{\chi^2}{\chi-1}}.$$

Altogether, we have obtained
$$\|\varphi_\varepsilon\|_{L^\infty(\omega)} \leq C(|\omega|, \kappa_+, \chi, r) \|f\|_{\mathcal{L}^p(\omega)} \left(\frac{\theta_1}{\mu_\varepsilon}\right)^{\theta_1} \left(\frac{\theta_2}{\lambda}\right)^{\theta_2}$$

For $p = 2$, we write $\theta_1 = \mu\frac{s}{2(s-2)}$, with $1 < \mu < \frac{2(s-2)}{s} < 2$ and use (79) to obtain
$$\|\varphi_\varepsilon\|_{L^\infty(\omega)} \leq C(\mu, s) \|f\|_{\mathcal{L}^2(\omega)} \frac{1}{\varepsilon^\mu \lambda^{1-\frac{\mu s}{2(s-2)}}}.$$

Now, for a fixed $1 < \alpha < 2$, we can choose $s > 2$ sufficient large so that we can choose $\mu = \alpha$ in the above estimate. By further enlarging s, we see that we have obtained the assertion for β as close to $1 - \frac{\alpha}{2}$ as we wish. The conclusion for smaller β also follows.
Case 2: $f = 0$. Using Hölder's inequality we get
$$B \leq \|\mathbf{h}\|_{\mathcal{L}^{2\tilde{p}}(\omega)} \|\nabla(w-k)_+\|_{\mathcal{L}^2(A(k))} N(A(k))^{\frac{1}{2}-\frac{1}{2\tilde{p}}}.$$

Together with (80), this shows that
$$\left\|(\varphi_\varepsilon - k)_+\right\|_{\mathcal{L}^r(A_k)} \leq \left(\frac{\theta_1}{\mu_\varepsilon}\right)^{\theta_1/2} \left(\frac{\theta_2}{\lambda}\right)^{\theta_2/2} \|\mathbf{h}\|_{\mathcal{L}^{2\tilde{p}}(\omega)} N(A(k))^{\chi/r}, \tag{82}$$

where $\frac{\chi}{r} = \frac{1}{2} - \frac{1}{2\tilde{p}}$. Note that (82) is a reverse Hölder inequality of the same type as (81). Therefore, arguing as above, and provided $\frac{1}{2} - \frac{1}{2\tilde{p}} > \frac{1}{r}$, to ensure that $\chi > 1$, we obtain
$$\|\varphi_\varepsilon\|_{L^\infty(\omega)} \leq C(|\omega|, \kappa_+, \chi, r) \|\mathbf{h}\|_{\mathcal{L}^{2\tilde{p}}(\omega)} \left(\frac{\theta_1}{\mu_\varepsilon}\right)^{\theta_1/2} \left(\frac{\theta_2}{\lambda}\right)^{\theta_2/2}$$

Considering now the case $\tilde{p} = \infty$, introducing $\theta_1 = (\alpha-1)\frac{s}{2(s-2)}$, with $1 < \alpha < 2\frac{s-2}{s}$, and $\beta = \frac{1-\theta_1}{2}$, we obtain
$$\|\varphi_\varepsilon\|_{L^\infty(\omega)} \leq C(\alpha, \beta) \|\mathbf{h}\|_{\mathcal{L}^{2\tilde{p}}(\omega)} \frac{1}{\varepsilon^{(\alpha-1)/2} \lambda^\beta},$$

as announced. $\quad\square$

Proof of Lemma 3. The proof is similar to the one above, with a_ε replaced by ν. The constant $\mu(s)$ is now independent of ε. Some of the details are provided in the proof of Lemma 1 for the reader's convenience. $\quad\square$

Lemma 4 *Assume that conditions (1), (2) and (3) hold. Then, for any $s \in [2, \infty)$, there exists a constant $C(s) > 0$ such that*

$$\int_\omega a_\varepsilon \, |v|^s \, dx \le C(s) \, \varepsilon^{2-s} \left(\int_\omega a_\varepsilon \, |\nabla v|^2 \, dx \right)^{\frac{s}{2}} \qquad \forall \, v \in H_0^1(\omega). \tag{83}$$

Proof As a first step of the proof, we establish a rescaled version of (83) in the cell $Y = (-1/2, 1/2)^2$. Define a rescaled conductivity $A_\varepsilon(y) = a_\varepsilon(\varepsilon y)$, for $y \in Y$. Using a r_ε-rescaling and the Sobolev embedding of $H^1(D_2)$ into $L^s(D_2)$ applied to the unit disk D_2, we get that

$$\forall \, V \in H^1(Y), \quad \fint_{D(r_\varepsilon)} \left| V - \fint_{D(r_\varepsilon)} V \right|^s dy \le C \, r_\varepsilon^s \left(\fint_{D(r_\varepsilon)} |\nabla V|^2 \, dy \right)^{\frac{s}{2}}.$$

Moreover, by estimate (3.13) in [11] we have

$$\forall \, V \in H^1(Y), \quad \int_Y \left| V - \fint_{D(r_\varepsilon)} V \right|^s dy \le C \, |\ln r_\varepsilon|^{\frac{s}{2}} \left(\int_Y |\nabla V|^2 \, dy \right)^{\frac{s}{2}}.$$

Then, combining the two previous estimates it follows that for any $V \in H^1(Y)$,

$$\fint_{D(r_\varepsilon)} \left| V - \int_Y V \right|^s dy \le C \left| \int_Y V \, dy - \fint_{D(r_\varepsilon)} V \, dy \right|^s + C \fint_{D(r_\varepsilon)} \left| V - \fint_{D(r_\varepsilon)} V \right|^s dy$$

$$\le C \, |\ln r_\varepsilon|^{\frac{s}{2}} \left(\int_Y |\nabla V|^2 \, dy \right)^{\frac{s}{2}} + C \, r_\varepsilon^s \left(\fint_{D(r_\varepsilon)} |\nabla V|^2 \, dy \right)^{\frac{s}{2}}.$$

Hence, by the definition of A_ε and (1),

$$\int_{D(r_\varepsilon)} A_\varepsilon \left| V - \int_Y V \right|^s dy \le C \, |\ln r_\varepsilon|^{\frac{s}{2}} \left(\int_Y A_\varepsilon \, |\nabla V|^2 \, dy \right)^{\frac{s}{2}}.$$

Since the Sobolev embedding inequality in Y gives

$$\int_Y \left| V - \int_Y V \right|^s dy \le C \left(\int_Y |\nabla V|^2 \, dy \right)^{\frac{s}{2}} \le C \left(\int_Y A_\varepsilon \, |\nabla V|^2 \, dy \right)^{\frac{s}{2}},$$

we thus deduce the following estimate in Y,

$$\forall \, V \in H^1(Y), \quad \int_Y A_\varepsilon \left| V - \int_Y V \right|^s dy \le C \, |\ln r_\varepsilon|^{\frac{s}{2}} \left(\int_Y A_\varepsilon \, |\nabla V|^2 \, dy \right)^{\frac{s}{2}}. \tag{84}$$

We now turn to the proof of (83). Let $v \in H_0^1(\omega)$ and extend it by zero outside ω. Rescaling estimate (84) in each square $\varepsilon \, (n + Y)$, for $n \in \mathbb{Z}^2$, we obtain that

$$\int_{\mathbb{R}^2} a_\varepsilon \, |v - \bar{v}_\varepsilon|^s \, dx \le C \, \varepsilon^2 \, |\ln r_\varepsilon|^{\frac{s}{2}} \sum_{n \in \mathbb{Z}^2} \left(\int_{\varepsilon(n+Y)} a_\varepsilon \, |\nabla v|^2 \, dx \right)^{\frac{s}{2}}$$

$$\le C \, \varepsilon^2 \, |\ln r_\varepsilon|^{\frac{s}{2}} \left(\int_{\mathbb{R}^2} a_\varepsilon \, |\nabla v|^2 \, dx \right)^{\frac{s}{2}} \quad \text{(since } s \ge 2\text{)},$$

where $\bar{v}_\varepsilon$ is the piecewise function which takes the average value of v in each square $\varepsilon(n+Y)$. Moreover, we have

$$
\begin{aligned}
\int_{\mathbb{R}^2} a_\varepsilon \left|\bar{v}_\varepsilon\right|^s dx &= \sum_{n\in\mathbb{Z}^2} \left(\fint_{\varepsilon(n+Y)} a_\varepsilon \, dx\right) \left|\fint_{\varepsilon(n+Y)} v \, dx\right|^s \\
&\leq \sum_{n\in\mathbb{Z}^2} C\,\varepsilon^2 \fint_{\varepsilon(n+Y)} |v|^s \, dx \quad \text{(by the Jensen inequality)} \\
&\leq C \int_\omega |v|^s \, dx \\
&\leq C \left(\int_\omega |\nabla v|^2 \, dx\right)^{\frac{s}{2}} \quad \text{(by the Sobolev embedding of } H_0^1(\omega) \text{ into } L^s(\omega)\text{).}
\end{aligned}
$$

Finally, combining the two previous inequalities we find

$$
\int_{\mathbb{R}^2} a_\varepsilon |v|^s \, dx \leq C\left(\varepsilon^2 \left|\ln r_\varepsilon\right|^{\frac{s}{2}} + 1\right) \left(\int_\omega a_\varepsilon |\nabla v|^2 \, dx\right)^{\frac{s}{2}},
$$

which yields the desired estimate (83) taking into account (1). $\quad\square$

Remark 8 The constant in the weighted inequality of [33] provides an estimate from above of the constant appearing in (83). It is not clear that this constant is optimal. The dependence in ε of the constant in (83) is optimal: this can be verified with the choice $v = g_\varepsilon$, where g_ε is given by (27).

7 Proof of Proposition 2

All of this section is in the two-dimensional setting. We will therefore drop the subscript 2 to denote two-dimensional gradients or divergences. We consider the solution u_ε of (21), for $\lambda \geq \lambda_0 > 0$,

$$
\begin{cases}
-\operatorname{div}(a_\varepsilon \nabla u_\varepsilon) + \lambda\, a_\varepsilon\, u_\varepsilon = f + a_\varepsilon\, g + \operatorname{div}(\mathbf{h}) & \text{in } \omega\,, \\
u_\varepsilon = 0 & \text{on } \partial\omega\,.
\end{cases}
$$

Let us start with a simple energy bound.

Lemma 5 *We have*

$$
\int_\omega a_\varepsilon \left[|\nabla u_\varepsilon|^2 + \lambda |u_\varepsilon|^2\right] dx \leq \frac{C}{\lambda} \int_\omega \left[|f|^2 + a_\varepsilon |g|^2\right] dx + C \int_\omega |\mathbf{h}|^2 \, dx. \tag{85}
$$

Proof Integrating (21) by parts against u_ε yields

$$
\int_\omega a_\varepsilon \left[|\nabla u_\varepsilon|^2 + \lambda |u_\varepsilon|^2\right] dx = \int_\omega \left[f u_\varepsilon + a_\varepsilon g u_\varepsilon - \mathbf{h}\cdot\nabla u_\varepsilon\right] dx
$$

Using the Cauchy-Schwarz inequality on the right-hand side we obtain (85). $\quad\square$

We now turn to the main part of our estimate. Our strategy, inspired by the limit case when the conductivity tends to infinity independently of the periodic structure [18,8,16], is to consider three contributions to u_ε. The first one is the contribution of the right-hand side when no highly conducting fibres are present. We introduce $v \in H_0^1(\omega)$, the solution to

$$
-\Delta v + \lambda v = f + g + \operatorname{div}(\mathbf{h}). \tag{86}
$$

The second is the contribution coming from the *average* of $u_\varepsilon - v$ on the cross section of the fibres. Set

$$u_{m,n} = \frac{1}{|\partial D_{m,n,\varepsilon}|} \int_{\partial D_{m,n,\varepsilon}} [u_\varepsilon(x) - v(x)] \, d\sigma(x),$$

and define $\tilde{u}_\varepsilon \in H^1(\omega \setminus D_\varepsilon)$ as the solution to

$$\begin{cases} -\Delta \tilde{u}_\varepsilon + \lambda \tilde{u}_\varepsilon = 0 & \text{in } \omega \setminus D_\varepsilon \,, \\ \tilde{u}_\varepsilon = 0 & \text{on } \partial\omega \,, \\ \tilde{u}_\varepsilon = u_{m,n} & \text{on } \partial D_{m,n,\varepsilon} \,. \end{cases} \tag{87}$$

The third is simply the remainder, given by $\hat{u}_\varepsilon = u_\varepsilon - v - \tilde{u}_\varepsilon$ on $\omega \setminus D_\varepsilon$. We will consider each contribution separately. Intuitively, our estimates show that, outside the fibres, u_ε consists "mainly" of two parts: v, the contribution of the background medium, and $\tilde{u}_\varepsilon$, the contribution from the high contrast.

Proof of Proposition 2. Fix $\tau > \kappa \varepsilon^{\frac{1-\eta}{2(1+\eta)}}$.

Since $\tilde{u}_\varepsilon + \hat{u}_\varepsilon = u_\varepsilon - v$, Lemma 7 (with $\beta = \eta/2$) and Corollary 4 show that

$$\|u_\varepsilon - v\|_{L^\infty(\omega_\tau^\varepsilon)} + \|\nabla u_\varepsilon - \nabla v\|_{L^\infty(\omega_\tau^\varepsilon)}$$
$$\leq \frac{C}{\lambda^{\frac{\eta}{2}}} \left(\frac{\varepsilon^2}{\lambda} \left(\|f\|_{L^2(\omega)} + \|\sqrt{a_\varepsilon} g\|_{L^2(\omega)} + \|\mathbf{h}\|_{L^\infty(\omega)} \right) + \|u_\varepsilon - v\|_{L^\infty(\omega')} \right)$$

where $\omega_0 \Subset \omega' \Subset \omega$. Using the L^∞ estimates on v given by Lemma 6 (with $q = 2/\eta$), we obtain

$$\|u_\varepsilon - v\|_{L^\infty(\omega_\tau^\varepsilon)} + \|\nabla u_\varepsilon - \nabla v\|_{L^\infty(\omega_\tau^\varepsilon)}$$
$$\leq \frac{C}{\lambda^{\frac{\eta}{2}}} \left(\frac{1}{\lambda^{\frac{\eta}{2}}} \left(\|f\|_{L^2(\omega)} + \|\sqrt{a_\varepsilon} g\|_{L^2(\omega)} + \|\mathbf{h}\|_{L^\infty(\omega)} \right) + \|u_\varepsilon\|_{L^\infty(\omega')} \right).$$

Lemma 10 together with Corollary 4 show that provided $\nu < \eta$,

$$[\nabla_2 u_\varepsilon - \nabla_2 v]_{C^\nu(\omega_\tau^\varepsilon)} \leq \frac{C}{\lambda^{\frac{\eta}{2} - \frac{\nu}{2}}} \|u_\varepsilon - v\|_{L^\infty(\omega')} + C \frac{\varepsilon^2}{\lambda^{\frac{3}{2}}} \left(\|f\|_{L^2(\omega)} + \|\sqrt{a_\varepsilon} g\|_{L^2(\omega)} + \|\mathbf{h}\|_{L^\infty(\omega)} \right).$$

Using again the L^∞ estimates on v given by Lemma 6, we obtain

$$[\nabla_2 u_\varepsilon - \nabla_2 v]_{C^\nu(\omega_\tau^\varepsilon)} \leq \frac{C}{\lambda^{\frac{\eta}{2} - \frac{\nu}{2}}} \left(\frac{1}{\lambda^{\frac{\eta}{2}}} \left(\|f\|_{L^2(\omega)} + \|\sqrt{a_\varepsilon} g\|_{L^2(\omega)} + \|\mathbf{h}\|_{L^\infty(\omega)} \right) + \|u_\varepsilon\|_{L^\infty(\omega')} \right),$$

which concludes the proof. $\square$

7.1 Estimates for v

It is quite straightforward to obtain estimates on v, as the following Lemma shows.

Lemma 6 *There holds*

$$\sqrt{\lambda} \|\nabla v\|_{L^2(\omega)} + \lambda \|v\|_{L^2(\omega)} \leq C \left[\|f\|_{L^2(\omega)} + \|g\|_{L^2(\omega)} + \sqrt{\lambda} \|\mathbf{h}\|_{L^2(\omega)} \right],$$

$$\|\nabla v\|_{L^q(\omega)} \leq C(q) \left[\|f\|_{L^2(\omega)} + \|g\|_{L^2(\omega)} + \|\mathbf{h}\|_{L^\infty(\omega)} \right], \qquad 2 \leq q < \infty.$$

Furthermore, for any $q > 2$, and any $\omega' \Subset \omega$,

$$\|v\|_{L^\infty(\omega')} \leq \frac{C(q, \omega')}{\lambda^{\frac{1}{q}}} \left[\|f\|_{L^2(\omega)} + \|g\|_{L^2(\omega)} + \|\mathbf{h}\|_{L^\infty(\omega)} \right].$$

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Proof The L^2 estimate for ∇v and v follows directly by testing (86) against v. To obtain the L^q gradient estimate, we write $v = v_1 + v_2$ where $v_1, v_2 \in H_0^1(\omega)$ are the solutions of

$$-\Delta v_1 = \lambda v + f + g = \tilde{f} \text{ and } -\Delta v_2 = \mathrm{div}(\mathbf{h}).$$

Next, by standard elliptic estimates, we have

$$\|v_1\|_{H^2(\omega)} \le C \|\tilde{f}\|_{L^2(\omega)} \le C\Big[\lambda \|v\|_{L^2(\omega)} + \|f+g\|_{L^2(\omega)}\Big] \le C\left(\|f\|_{L^2(\omega)} + \|g\|_{L^2(\omega)}\right),$$
$$\|v_2\|_{W^{1,q}(\omega)} \le C \|\mathbf{h}\|_{L^q(\omega)}.$$

Since $H^2(\omega) \hookrightarrow W^{1,q}(\omega)$, the second estimate in the lemma follows. The last assertion follows from Lemma 3. □

A direct consequence of the above result is a local L^2 bound on ∇v.

Corollary 3 *For any $\omega' \Subset \omega$, $\beta < \frac{1}{2}$ and $\varrho < \frac{1-2\beta}{4}$, there holds*

$$\|\nabla v\|_{L^2(\omega')} \le C |\omega'|^\varrho \left[\frac{1}{\lambda^\beta}\left(\|f\|_{L^2(\omega)} + \|\sqrt{a_\varepsilon}\, g\|_{L^2(\omega)}\right) + \|\mathbf{h}\|_{L^\infty(\omega)}\right].$$

Proof We split $v = v_a + v_b$ where $v_a, v_b \in H_0^1(\omega)$ are the solutions to

$$-\Delta v_a + \lambda v_a = f + g \text{ and } -\Delta v_b + \lambda v_b = \mathrm{div}(\mathbf{h}).$$

By the L^q gradient bound in Lemma 6 and the Sobolev embedding theorem,

$$\|\nabla v_a\|_{L^p(\omega)} \le C\left(\|f\|_{L^2(\omega)} + \|g\|_{L^2(\omega)}\right) \text{ and } \|\nabla v_b\|_{L^p(\omega)} \le C\|\mathbf{h}\|_{L^\infty(\omega)} \text{ for any } p > 2.$$

Thus, by Hölder's inequality, we have, for any $\omega' \subset \omega$ and $\delta \in (0,1)$,

$$\|\nabla v_a\|_{L^2(\omega')} \le C\left(\|f\|_{L^2(\omega)} + \|g\|_{L^2(\omega)}\right) |\omega'|^{1-\delta} \text{ and } \|\nabla v_b\|_{L^2(\omega')} \le C\|\mathbf{h}\|_{L^\infty(\omega)} |\omega'|^{1-\delta}.$$

On the other hand, by Lemma 6, we also have

$$\|\nabla v_a\|_{L^2(\omega)} \le \frac{C}{\sqrt{\lambda}}\left(\|f\|_{L^2(\omega)} + \|g\|_{L^2(\omega)}\right) \text{ and } \|\nabla v_b\|_{L^2(\omega)} \le C\|\mathbf{h}\|_{L^2(\omega)}.$$

The conclusion follows by an interpolation. □

7.2 Estimates for $\tilde{u}_\varepsilon$

The first part of our estimate for $\tilde{u}_\varepsilon$ is given by the following Lemma. The second part, concerning the Hölder regularity of its gradient, is given by Lemma 10.

Lemma 7 *For any $0 < \beta < \frac{1}{2}$ and $\tau > \kappa\varepsilon^{\frac{1-2\beta}{1+2\beta}}$, the solution $\tilde{u}_\varepsilon$ of (87) satisfies*

$$\|\tilde{u}_\varepsilon\|_{L^\infty(\omega_\tau^\varepsilon)} + \lambda^{\beta-1/2}\|\nabla \tilde{u}_\varepsilon\|_{L^\infty(\omega_\tau^\varepsilon)} \le \frac{C}{\lambda^{1/2}} \sup_{m,n} |u_{m,n}|,$$

where the constant C depends on β, κ and ω only, and ω_τ^ε is defined in (9).

Here and below $\sup_{m,n}$ refers to the supremum taken as (m,n) varies in I_ε.
We shall use two local estimations. We first estimate the gradient $\nabla \tilde{u}_\varepsilon$ on the outer boundary $\partial\omega$.

Lemma 8 *There holds*

$$\|\nabla \tilde{u}_\varepsilon\|_{L^\infty(\partial\omega)} \leq C \sup_{m,n} |u_{m,n}|.$$

Proof By the maximum principle,

$$\sup_{\omega\setminus D_\varepsilon} |\tilde{u}_\varepsilon| \leq \sup_{m,n} |u_{m,n}| . \tag{88}$$

Fix $\omega_0 \Subset \omega' \Subset \omega'' \Subset \omega$. Let η be a cut-off function which is one in $\omega \setminus \omega'$ and vanishes in ω_0. Using $\eta^2 \tilde{u}_\varepsilon$ as a test function in (87), we obtain

$$\int_{\omega\setminus\omega_0} \eta^2 |\nabla\tilde{u}_\varepsilon|^2 + \int_{\omega\setminus\omega_0} \lambda\eta^2\tilde{u}_\varepsilon^2 = -2\int_{\omega\setminus\omega_0} \eta\tilde{u}_\varepsilon\nabla\eta\cdot\nabla\tilde{u}_\varepsilon,$$

$$\leq \frac{1}{2}\int_{\omega\setminus\omega_0} \eta^2|\nabla\tilde{u}_\varepsilon|^2 + 2\int_{\omega\setminus\omega_0} u_\varepsilon^2\,|\nabla\eta|^2.$$

Thanks to (88), we conclude that

$$\|\nabla\tilde{u}_\varepsilon\|_{L^2(\omega\setminus\omega')} \leq C \sup_{m,n} |u_{m,n}| . \tag{89}$$

Let $u^\pm \in H^1(\omega \setminus \omega')$ be the solution to

$$\begin{cases} -\Delta u^\pm = 0 & \text{in } \omega \setminus \omega' , \\ u^\pm = 0 & \text{on } \partial\omega , \\ u^\pm = \tilde{u}_\varepsilon^\pm & \text{on } \partial\omega' , \end{cases}$$

where $\tilde{u}_\varepsilon^\pm = \max(\pm\tilde{u}_\varepsilon, 0)$. By H^2-estimates, valid up the boundary $\partial\omega$ thanks to the vanishing boundary condition and the regularity of $\partial\omega$, $u^\pm \in H^2(\omega\setminus\omega'')$, with

$$\|u^\pm\|_{H^2(\omega\setminus\omega'')} \leq C\,\|\tilde{u}_\varepsilon\|_{H^1(\omega\setminus\omega')}.$$

This, in turn, because $u^\pm$ satisfies a homogeneous equation in $\omega \setminus \omega'$, implies a better estimate, namely a $W^{2,p}$-estimates for an arbitrary p [20, Lemma 9.16]. For p large enough, thanks to the Sobolev embedding Theorem, $u^\pm \in W^{2,p}(\omega\setminus\omega'') \hookrightarrow W^{1,\infty}(\omega\setminus\omega'')$ and we deduce that

$$\|u^\pm\|_{W^{1,\infty}(\omega\setminus\omega'')} \leq C\,\|\tilde{u}_\varepsilon\|_{H^1(\omega\setminus\omega')} .$$

Inserting (88) and (89) in this last estimate, we obtain

$$\|u^\pm\|_{W^{1,\infty}(\omega\setminus\omega'')} \leq C \sup_{m,n} |u_{m,n}| . \tag{90}$$

As an application of the maximum principle we note that $u^\pm \geq 0$ and so

$$-\Delta u^+ + \lambda\,u^+ \geq 0 = -\Delta\tilde{u}_\varepsilon + \lambda\tilde{u}_\varepsilon \geq -\Delta(-u^-) + \lambda\,(-u^-) \text{ in } \omega \setminus \omega' .$$

Applying the maximum principle again, we thus get

$$u^+ \geq \tilde{u}_\varepsilon \geq -u^- \text{ in } \omega \setminus \omega'.$$

Since the three functions agree on $\partial\omega$, the required estimate follows from (90). □

Next, we estimate the trace of the gradient $\nabla\tilde{u}_\varepsilon$ on the boundary of each rod $D_{m,n,\varepsilon}$.

Lemma 9 *There holds*

$$\|\nabla \tilde{u}_\varepsilon\|_{L^\infty(\partial D_{m,n,\varepsilon})} \le C\theta(\lambda,\varepsilon) \sup_{m,n} |u_{m,n}| \,,$$

where

$$\theta(\lambda,\varepsilon) = \begin{cases} \dfrac{\varepsilon}{r_\varepsilon} & \text{if } \sqrt{\lambda} \le \dfrac{1}{\varepsilon}, \\[2ex] \dfrac{\varepsilon}{r_\varepsilon(1-\alpha+\varepsilon^2)} & \text{if } \sqrt{\lambda} = \dfrac{1}{\varepsilon r_\varepsilon^\alpha} \text{ with } 0 \le \alpha \le 1, \\[2ex] \sqrt{\lambda} & \text{if } \dfrac{1}{\varepsilon r_\varepsilon} \le \sqrt{\lambda}. \end{cases} \tag{91}$$

This result is proved in Appendix A.3.

We are now in position to prove Lemma 7. Note that the fundamental solution Φ of $-\Delta + \lambda$ on $\mathbb{R}^2$ is given by

$$\Phi(x) = \Phi(|x|) = \frac{1}{2\pi} K_0(\sqrt{\lambda}\,|x|) \,, \tag{92}$$

where K_0 is the second modified Bessel function of order zero. In what follows we will make extensive use of well known facts concerning K_0 and its derivatives, easily recovered from the recurrence relations and asymptotic properties, see e.g. [32]. Let $K_0^{(n)}$ denote de n-th derivative of K_0, $n \ge 0$. Then $|K_0^{(n)}|$ is decreasing, and, for all $x > 0$ and $\alpha \ge 0$,

$$|K_0(x)| \le C(\alpha)\frac{|\ln(x)|}{1+x^\alpha} \text{ and } |K_0^{(n)}(x)| \le \frac{C(n,\alpha)}{x^n(1+x^\alpha)}, n \ge 1. \tag{93}$$

Proof of Lemma 7. Our argument uses layer potentials. We set $M = \sup_{m,n} |u_{m,n}|$. Let Φ be given by (92). We have

$$\tilde{u}_\varepsilon(x) = \int_{\partial(\omega \setminus D_\varepsilon)} \left[\frac{\partial \tilde{u}_\varepsilon}{\partial n(y)}(y)\,\Phi(x-y) - \tilde{u}_\varepsilon(y)\,\frac{\partial \Phi}{\partial n(y)}(x-y) \right] d\sigma(y) \,. \tag{94}$$

Here $n(y)$ denotes the outward unit normal to $\partial(\omega \setminus D_\varepsilon)$ at y. Fix any multi-index $J \in \mathbb{Z}^2$. Differentiating (94) we obtain

$$\begin{aligned} \nabla^J \tilde{u}_\varepsilon(x) &= \int_{\partial(\omega \setminus D_\varepsilon)} \left[\frac{\partial \tilde{u}_\varepsilon}{\partial n(y)}(y)\,\nabla_x^J \Phi(x-y) - \tilde{u}_\varepsilon(y)\,\frac{\partial \nabla_x^J \Phi}{\partial n(y)}(x-y) \right] d\sigma(y) \\ &= I_1^J + I_2^J + I_3^J \,, \end{aligned} \tag{95}$$

where

$$I_1^J = \int_{\partial \omega} \frac{\partial \tilde{u}_\varepsilon}{\partial n(y)}(y)\,\nabla_x^J \Phi(x-y)\,d\sigma(y),$$

$$I_2^J = -\sum_{(m,n) \in I_\varepsilon} \int_{\partial D_{m,n,\varepsilon}} \tilde{u}_\varepsilon(y)\,\frac{\partial \nabla_x^J \Phi}{\partial n(y)}(x-y)\,d\sigma(y),$$

$$I_3^J = \sum_{(m,n) \in I_\varepsilon} \int_{\partial D_{m,n,\varepsilon}} \frac{\partial \tilde{u}_\varepsilon}{\partial n(y)}(y)\,\nabla_x^J \Phi(x-y)\,d\sigma(y) \,.$$

Fix $x \in \omega_\tau^\varepsilon$. We now estimate the right-hand side of (95). We begin with the easy estimates for I_1^J and I_2^J. Thanks to (93) and Lemma 8,

$$|I_1^J| \leq \frac{C(|J|, \tilde{\beta})}{\lambda^{\tilde{\beta}}} M \text{ for all } \tilde{\beta} \geq 0. \tag{96}$$

To bound I_2^J, we estimate the contributions of inclusions located "radially" between $\varepsilon\tau + j\varepsilon$ and $\varepsilon\tau + (j+1)\varepsilon$ for $j = 0, 1, \ldots$ We obtain using (93),

$$|I_2^J| \leq C \sum_{j=0}^{C\varepsilon^{-1}} M\,(j+1)\,\sqrt{\lambda}^{|J|+1} \left| K_0^{(|J|+1)}(\sqrt{\lambda}\,(j+\tau)\varepsilon) \right| \varepsilon\, r_\varepsilon$$

$$\leq CM\, \frac{r_\varepsilon}{\lambda^{\tilde{\beta}}\,\tau^{|J|+1+2\tilde{\beta}}\,\varepsilon^{|J|+2\tilde{\beta}}} \sum_{j=0}^{C\varepsilon^{-1}} \left(\sqrt{\lambda}\,(j+\tau)\,\varepsilon\right)^{|J|+1+2\tilde{\beta}} \left| K_0^{(|J|+1)}(\sqrt{\lambda}\,(j+\tau)\varepsilon) \right|$$

$$\leq CM\, \frac{r_\varepsilon}{\lambda^{\tilde{\beta}}\,\tau^{|J|+1+2\tilde{\beta}}\,\varepsilon^{|J|+1+2\tilde{\beta}}} \sup_{0 < t < \infty} t^{|J|+1+2\tilde{\beta}} K_0^{(|J|+1)}(t)$$

$$= CM\, \frac{r_\varepsilon}{\lambda^{\tilde{\beta}}\,\tau^{|J|+1+2\tilde{\beta}}\,\varepsilon^{|J|+1+2\tilde{\beta}}}, \qquad \tilde{\beta} \geq 0. \tag{97}$$

Let us now turn to the third integral, I_3^J. Using Lemma 9 and counting contributions from "rings" of inclusions as above, we have

$$|I_3^J| \leq C \sum_{j=0}^{C\varepsilon^{-1}} \theta(\lambda, \varepsilon)\, M\,(j+1)\,\sqrt{\lambda}^{|J|} \left| K_0^{(|J|)}(\sqrt{\lambda}\,(j+\tau)\varepsilon) \right| \varepsilon\, r_\varepsilon.$$

Since $\theta(\lambda, \varepsilon)$ can be "large", we proceed to decompose I_3^J into two parts: one counts the contribution from the ring of inclusions closest to x, where the dependence on τ is dominant, and the other counts the contribution of the further away inclusions, where τ does not play a role. Proceeding in this way, we compute, using the decay properties of $K_0^{(|J|)}$ for the second term,

$$|I_3^J| \leq C \sum_{j=0}^{C\varepsilon^{-1}} \theta(\lambda, \varepsilon)\, M\,(j+1)\,\sqrt{\lambda}^{|J|} \left| K_0^{(|J|)}(\sqrt{\lambda}\,(j+\tau)\varepsilon) \right| \varepsilon\, r_\varepsilon$$

$$\leq \underbrace{C\,M\,\theta(\lambda, \varepsilon)\,\varepsilon\, r_\varepsilon\, \sqrt{\lambda}^{|J|} \left| K_0^{(|J|)}(\sqrt{\lambda}\,\tau\,\varepsilon) \right|}_{X^J = \text{near neighbour terms, i.e. } j = 0}$$

$$+ \underbrace{C\,M\,\theta(\lambda, \varepsilon)\,\varepsilon\, r_\varepsilon \sum_{j=1}^{C\varepsilon^{-1}} j\,\sqrt{\lambda}^{|J|} \left| K_0^{(|J|)}(\sqrt{\lambda}\,j\,\varepsilon) \right|}_{Y^J = \text{distant neighbour terms}}.$$

The near term X^J can be bounded as follows,

$$X^J \leq C\,M\, \frac{\theta(\lambda, \varepsilon)\, r_\varepsilon}{\lambda^{\tilde{\beta}}\,\tau^{|J|+2\tilde{\beta}}\,\varepsilon^{|J|-1+2\tilde{\beta}}} \left(\sqrt{\lambda}\,\tau\,\varepsilon\right)^{|J|+2\tilde{\beta}} \left| K_0^{(|J|)}(\sqrt{\lambda}\,\tau\,\varepsilon) \right|$$

$$\leq C\,M\, \frac{\theta(\lambda, \varepsilon)\, r_\varepsilon}{\lambda^{\tilde{\beta}}\,\tau^{|J|+2\tilde{\beta}}\,\varepsilon^{|J|-1+2\tilde{\beta}}} \sup_{t > 0} t^{|J|+2\tilde{\beta}} \left| K_0^{(|J|)}(t) \right|$$

$$\leq C\,M\, \frac{\theta(\lambda, \varepsilon)\, r_\varepsilon}{\lambda^{\tilde{\beta}}\,\tau^{|J|+2\tilde{\beta}}\,\varepsilon^{|J|-1+2\tilde{\beta}}}, \qquad \tilde{\beta} > 0. \tag{98}$$

Similarly, we have

$$Y^J \leq C\,M\,\frac{\theta(\lambda,\varepsilon)\,r_\varepsilon}{\lambda^{\tilde\beta}\,\varepsilon^{|J|-1+2\tilde\beta}}\sum_{j=1}^{C\varepsilon^{-1}}(\sqrt{\lambda}\,j\,\varepsilon)^{|J|+2\tilde\beta}\left|K_0^{(|J|)}(\sqrt{\lambda}\,j\,\varepsilon)\right|$$

$$\leq C\,M\,\frac{\theta(\lambda,\varepsilon)\,r_\varepsilon}{\lambda^{\tilde\beta}\,\varepsilon^{|J|+2\tilde\beta}}\sup_{t>0}t^{|J|+2\tilde\beta}\left|K_0^{(|J|)}(t)\right|$$

$$\leq C\,M\,\frac{\theta(\lambda,\varepsilon)\,r_\varepsilon}{\lambda^{\tilde\beta}\,\varepsilon^{|J|+2\tilde\beta}},\qquad \tilde\beta>0. \tag{99}$$

We then insert estimate (91) in (98) and (99). For $\lambda \geq \frac{1}{r_\varepsilon}$, we can choose $\tilde\beta=1$ and obtain

$$|I_3^J| \leq C\,M\,\max\left(\frac{1}{\varepsilon\,r_\varepsilon},\sqrt{\lambda}\right)\frac{r_\varepsilon}{\lambda\,\varepsilon^{|J|+2}}\left(\frac{\varepsilon}{\tau^{|J|+2}}+1\right)$$

$$\leq C\,M\,\max\left(\frac{1}{\varepsilon\,r_\varepsilon^{1/2}},1\right)\frac{r_\varepsilon}{\sqrt{\lambda}\,\varepsilon^{|J|+2}}\left(\frac{\varepsilon}{\tau^{|J|+2}}+1\right)$$

$$\leq C\,M\,\frac{r_\varepsilon^{1/2}}{\sqrt{\lambda}\,\varepsilon^{|J|+3}}\left(\frac{\varepsilon}{\tau^{|J|+2}}+1\right). \tag{100}$$

For $\lambda \leq \frac{1}{r_\varepsilon}$ we can take $\tilde\beta=\frac12$ to get

$$|I_3^J| \leq C\,M\,\frac{1}{\sqrt{\lambda}\,\varepsilon^{|J|}}\left(\frac{\varepsilon}{\tau^{|J|+1}}+1\right). \tag{101}$$

These estimates are enough to conclude for the case $J=(0,0)$. Adding (96), (97) (with $\tilde\beta=\frac12$), (100) and (101), we obtain, for all $\lambda \geq \lambda_0$,

$$|\tilde u_\varepsilon(x)| \leq \frac{CM}{\sqrt{\lambda}}\left(1+\frac{r_\varepsilon}{\tau^2\,\varepsilon^2}+\frac{\varepsilon}{\tau}+\frac{r_\varepsilon^{1/2}}{\tau^2\,\varepsilon^3}\left(\frac{\varepsilon}{\tau^2}+1\right)\right),$$

which proves the first desired estimate as $\tau > \kappa\,\varepsilon^{\frac{1-2\beta}{1+2\beta}} \gg \varepsilon$.

For $|J|=1$ and $\lambda \leq \frac{1}{r_\varepsilon}$, estimate (101) deteriorates. We keep the near estimate (98) for X^J and improve the far estimate (99) for Y^J. First, we write

$$Y^{(0,1)}+Y^{(1,0)} = C\,M\,\theta(\lambda,\varepsilon)\,\varepsilon\,r_\varepsilon\sum_{j=1}^{C\varepsilon^{-1}}j\,\sqrt{\lambda}\left|K_0^{(1)}(\sqrt{\lambda}\,j\,\varepsilon)\right|$$

$$\leq C\,M\,\frac{\theta(\lambda,\varepsilon)\,r_\varepsilon}{\varepsilon}\sum_{j=1}^{C\varepsilon^{-1}}(\sqrt{\lambda}\,j\,\varepsilon)\left|K_0^{(1)}(\sqrt{\lambda}\,j\,\varepsilon)\right|\varepsilon.$$

Note that since $t\to t\,|K_0^{(1)}(t)|$ is decreasing and summable on $(0,\infty)$, its lower Riemann sum is bounded from above by the continuous integral. Thus

$$Y^{(0,1)}+Y^{(1,0)} \leq C\,M\,\frac{\theta(\lambda,\varepsilon)\,r_\varepsilon}{\varepsilon}\int_0^\infty \sqrt{\lambda}t\,K_0^{(1)}(\sqrt{\lambda}t)\,dt$$

$$= C\,M\,\frac{\theta(\lambda,\varepsilon)\,r_\varepsilon}{\sqrt{\lambda}\,\varepsilon}.$$

This inequality together with (91) and (98) yields that for $\lambda \leq \frac{1}{r_\varepsilon}$,

$$|I_3^{(1,0)}| + |I_3^{(0,1)}| \leq \frac{CM}{\lambda^{\tilde{\beta}}} \Big(\frac{\varepsilon^{1-2\tilde{\beta}}}{\tau^{1+2\tilde{\beta}}} + 1 \Big), \qquad 0 < \tilde{\beta} < \frac{1}{2}. \tag{102}$$

Choosing $\tilde{\beta} = \beta$ in the above estimate and in (96), (97) and recalling (100), we conclude for all $\lambda \geq \lambda_0$ that

$$|\nabla \tilde{u}_\varepsilon(x)| \leq \frac{CM}{\lambda^\beta} \Big(1 + \frac{r_\varepsilon}{\tau^{2+2\beta}} \frac{1}{\varepsilon^{2+2\beta}} + \frac{\varepsilon^{1-2\beta}}{\tau^{1+2\beta}} + \frac{r_\varepsilon^{1/2}}{\varepsilon^4} \Big(\frac{\varepsilon}{\tau^3} + 1 \Big) \Big),$$

which provides the remaining estimate as $\tau > \kappa \varepsilon^{\frac{1-2\beta}{1+2\beta}}$. □

We now turn to the derivation of Hölder estimates for the gradient of $\tilde{u}_\varepsilon$.

Lemma 10 *For any $0 < \beta < \frac{1}{2}$, $0 < \nu < 2\beta$, $\kappa > 0$ and $\tau > \kappa \varepsilon^{\frac{1-2\beta}{2(1+2\beta-\nu)}}$ there holds*

$$[\nabla \tilde{u}_\varepsilon]_{C^\nu(\omega_\tau^\varepsilon)} \leq \frac{C}{\lambda^{\beta - \frac{\nu}{2}}} \sup_{m,n} |u_{m,n}|$$

for some constant C independent of ε, τ and λ.

This Lemma is proved in Appendix A.4.

7.3 Estimates for $\hat{u}_\varepsilon$

The residual term $\hat{u}_\varepsilon$ satisfies

$$\begin{cases} -\Delta \hat{u}_\varepsilon + \lambda \hat{u}_\varepsilon = 0 & \text{in } \omega \setminus D_\varepsilon \,, \\ \hat{u}_\varepsilon = u_\varepsilon - v - u_{m,n} & \text{on } \partial D_{m,n,\varepsilon} \,, (m,n) \in I_\varepsilon \,. \end{cases} \tag{103}$$

We derive the following estimate, which shows that $\hat{u}_\varepsilon$ is negligible compared to other contributions to u_ε.

Lemma 11 *For any $J \in \mathbb{Z}^2$, $\eta > 0$, and $\tau > \kappa \varepsilon^{-\frac{2+|J|+2\eta}{|J|+2\eta}} r_\varepsilon^{\frac{1}{16(|J|+2\eta)}}$, the solution $\hat{u}_\varepsilon$ of (103) satisfies*

$$\|\nabla^J \hat{u}_\varepsilon\|_{L^\infty(\omega_\tau^\varepsilon)} \leq \frac{C \, r_\varepsilon^{\frac{1}{16}}}{\lambda^\eta} \big[\|f\|_{L^2(\omega)} + \|\sqrt{a_\varepsilon} g\|_{L^2(\omega)} + \|\mathbf{h}\|_{L^\infty(\omega)} \big],$$

where the constant C depends on J, η, κ and ω only, and ω_τ^ε is defined in (9).

For the proof of Proposition 2, we shall use the following simple consequence of Lemma 11.

Corollary 4 *For any $\tau > \kappa \varepsilon^2$, the solution $\hat{u}_\varepsilon$ of (103) satisfies*

$$\|\hat{u}_\varepsilon\|_{L^\infty(\omega_\tau^\varepsilon)} + \|\nabla \hat{u}_\varepsilon\|_{L^\infty(\omega_\tau^\varepsilon)} + [\nabla \hat{u}_\varepsilon]_{C^{0,1}(\overline{\omega_\tau^\varepsilon})} \leq \frac{C \, \varepsilon^2}{\lambda^{\frac{3}{2}}} \big[\|f\|_{L^2(\omega)} + \|\sqrt{a_\varepsilon} g\|_{L^2(\omega)} + \|\mathbf{h}\|_{L^\infty(\omega)} \big],$$

where $C > 0$ depends on κ and ω only, and ω_τ^ε is defined in (9).

Remark 9 A similar statement is possible with $\frac{\varepsilon^2}{\lambda^{\frac{3}{2}}}$ replaced by $\frac{\varepsilon^\alpha}{\lambda^\beta}$ with arbitrary $\alpha > 0$ and $\beta > 0$. However, we chose those powers to fix ideas, since, for example, $\|v\|_{L^\infty(\omega)}$ and $\|\tilde{u}_\varepsilon\|_{L^\infty(\omega_\tau^\varepsilon)}$ have slower decay in λ, and the restriction on τ (with respect to ε) is more stringent.

The proof will use the following result, whose proof will be given later.

Lemma 12 *Given $\alpha \in H^{1/2}\left(\mathbb{S}^1\right)$, and $s > 0$, there exists a unique solution in $H^1(\mathbb{R}^2 \setminus B(0,1))$ on the exterior of the unit disk of $\mathbb{R}^2$ of*

$$-\Delta u + su = 0 \text{ in } \mathbb{R}^2 \setminus B(0,1) \quad u = \alpha \text{ on } \mathbb{S}^1,$$

still denoted by α, which decays at infinity. It satisfies

$$|\alpha(x)| \leq C\, e^{-\sqrt{s}\,|x|/4}\, \|\alpha\|_{L^2(\mathbb{S}^1)} \text{ for } |x| \geq 3. \tag{104}$$

Remark 10 We do not claim that this estimate is in any way optimal, but is it sufficient for our purpose.

Proof of Lemma 11. We first derive a bound for $\hat{u}_\varepsilon$ away from D_ε. For a given $(m,n) \in I_\varepsilon$, let $\alpha_{m,n}^+$ be the solution to

$$\begin{cases} -\Delta \alpha_{m,n}^+ + \lambda\, \alpha_{m,n}^+ = 0 & \text{in } \mathbb{R}^2 \setminus D_{m,n,\varepsilon}, \\ \alpha_{m,n}^+ = \hat{u}_\varepsilon^+ & \text{on } \partial D_{m,n,\varepsilon}, \\ \alpha_{m,n}^+(x') \to 0 & \text{as } |x'| \to \infty. \end{cases}$$

By the maximum principle, $\alpha_{m,n}^+$ is non-negative in $\omega \setminus D_\varepsilon$. Thus, again by the maximum principle,

$$\hat{u}_\varepsilon \leq \sum_{(m,n) \in I_\varepsilon} \alpha_{m,n}^+ \text{ in } \omega \setminus D_\varepsilon. \tag{105}$$

Now, thanks to Lemma 12 we have

$$|\alpha_{m,n}^+| \leq C \exp\left(-\frac{\sqrt{\lambda}\,\varepsilon\,\tau}{16}\right) \frac{1}{\sqrt{\varepsilon\,r_\varepsilon}} \|u_\varepsilon - v - u_{m,n}\|_{L^2(\partial D_{m,n,\varepsilon})} \text{ in } \omega_{\tau/4}^\varepsilon.$$

On the other hand, by Poincaré's inequality, (85) and Corollary 3,

$$\|u_\varepsilon - v - u_{m,n}\|_{L^2(\partial D_{m,n,\varepsilon})} \leq C\,\sqrt{\varepsilon\,r_\varepsilon} \left[\|\nabla u_\varepsilon\|_{L^2(D_{m,n,\varepsilon})} + \|\nabla v\|_{L^2(D_{m,n,\varepsilon})}\right]$$
$$\leq C\,\sqrt{\varepsilon\,r_\varepsilon}\,\left(r_\varepsilon + (\varepsilon\,r_\varepsilon)^{2\varrho}\right) \times$$
$$\times \underbrace{\left[\frac{1}{\lambda^\beta}\,\left(\|f\|_{L^2(\omega)} + \|\sqrt{a_\varepsilon}\,g\|_{L^2(\omega)}\right) + \|\mathbf{h}\|_{L^\infty(\omega)}\right]}_{=:\zeta}$$

for $\beta = \frac{1}{4} \in (0, \frac{1}{2})$ and $\varrho = \frac{1}{16} \in (0, \frac{1-2\beta}{4})$. Inserting this estimate in the upper bound of $\alpha_{m,n}^+$ gives

$$\alpha_{m,n}^+ \leq C \exp\left(-\frac{\sqrt{\lambda}\,\varepsilon\,\tau}{16}\right)\left(r_\varepsilon + (\varepsilon\,r_\varepsilon)^{2\varrho}\right)\zeta \text{ in } \omega_{\tau/4}^\varepsilon.$$

Substituting the above estimate into (105) results in

$$\hat{u}_\varepsilon \leq C \exp\left(-\frac{\sqrt{\lambda}\,\varepsilon\,\tau}{16}\right)\frac{\left(r_\varepsilon + (\varepsilon\,r_\varepsilon)^{2\varrho}\right)}{\varepsilon^2}\zeta \text{ in } \omega_{\tau/4}^\varepsilon.$$

A similar lower bound is derived by repeating the argument. We have obtained

$$\|\hat{u}_\varepsilon\|_{L^\infty(\omega_{\tau/4}^\varepsilon)} \leq C \exp\left(-\frac{\sqrt{\lambda}\,\varepsilon\,\tau}{16}\right)\frac{\left(r_\varepsilon + (\varepsilon\,r_\varepsilon)^{2\varrho}\right)}{\varepsilon^2}\zeta. \tag{106}$$

We now proceed to prove the required derivative estimate for $\hat{u}_\varepsilon$.

We will only bound $\xi = \partial_{x_1}\hat{u}_\varepsilon$. The other (higher) partial derivatives can be bounded similarly. Fix $x \in \omega_\tau^\varepsilon$. Note that the disk $B_{3\varepsilon\tau/4}(x) \subset \omega_{\tau/4}^\varepsilon$. Using a suitable cut-off function which is one in $B_{\varepsilon\tau/2}(x)$ and zero outside $B_{3\varepsilon\tau/4}(x)$ and a standard energy estimate, it is easy to show that

$$\|\nabla \hat{u}_\varepsilon\|_{L^2(B_{\varepsilon\tau/2}(x_0))} \le \frac{C}{\varepsilon\tau}\|\hat{u}_\varepsilon\|_{L^2(B_{3\varepsilon\tau/4}(x_0)\setminus B_{\varepsilon\tau/2}(x_0))}$$

which together with (106) implies

$$\|\nabla \hat{u}_\varepsilon\|_{L^2(B_{\varepsilon\tau/2}(x))} \le C \exp\left(-\frac{\sqrt{\lambda}\,\varepsilon\,\tau}{16}\right) \frac{\left(r_\varepsilon + (\varepsilon\,r_\varepsilon)^{2\varrho}\right)}{\varepsilon^2}\,\zeta. \tag{107}$$

On the other hand, by differentiating (103),

$$-\Delta\xi + \lambda\,\xi = 0 \text{ in } \omega \setminus D_\varepsilon .$$

Usual interior De Giorgi estimates then show that

$$\|\xi\|_{L^\infty(B_{\varepsilon\tau/4}(x))} \le \frac{C}{\varepsilon\tau}\,\|\xi\|_{L^2(B_{\varepsilon\tau/2}(x))}.$$

Combining this bound with (107) and noting that x is arbitrary in ω_τ^ε, we get

$$\|\xi\|_{L^\infty(\omega_\tau^\varepsilon)} \le C \exp\left(-\frac{\sqrt{\lambda}\,\varepsilon\,\tau}{16}\right) \frac{\left(r_\varepsilon + (\varepsilon\,r_\varepsilon)^{2\varrho}\right)}{\tau\varepsilon^3}\,\zeta.$$

The required estimate for ξ then follows from the simple inequality $e^{-|x|} \le \frac{C(\eta)}{|x|^\eta}$. Further higher derivative estimates are obtained by repeating the above process. $\square$

To conclude, we now provide the proof of Lemma 12.

Proof of Lemma 12. It is straightforward to check that if $\alpha \in L^2\left(\mathbb{S}^1\right)$ has the Fourier expansion

$$\alpha(\theta) = \sum_{k=0}^\infty \left[a_k \cos k\theta + b_k \sin k\theta\right],$$

then it extends to a solution of $(-\Delta + s)\alpha = 0$ on $\mathbb{R}^2 \setminus D_1$ which vanishes at infinity by

$$\alpha(r,\theta) = \sum_{k=0}^\infty \frac{1}{K_k(\sqrt{s})}\left[a_k \cos k\theta + b_k \sin k\theta\right]K_k(\sqrt{s}\,r),$$

where K_k is the second modified Bessel Function of order k. The function K_k has a representation as follows (see e.g. [32])

$$K_k(t) = \int_0^\infty e^{-t\cosh\tau}\,\cosh k\tau\,d\tau .$$

This implies the obvious bound

$$\frac{K_k(\sqrt{s}\,r)}{K_k(\sqrt{s})} \le e^{-\sqrt{s}(r-1)}, \qquad k \ge 0.$$

On the other hand, for $k \geq 1$, we also have

$$K_k(t) \leq \int_0^\infty e^{-\frac{1}{2} t e^\tau} e^{k\tau} d\tau = \frac{2^k}{t^k} \int_0^\infty e^{-\tau} \tau^{k-1} d\tau = \frac{2^k (k-1)!}{t^k} ,$$

and

$$K_k(t) \geq \frac{1}{2} \int_0^\infty e^{-t e^\tau} e^{k\tau} d\tau = \frac{1}{2 t^k} \int_0^\infty e^{-\tau} \tau^{k-1} d\tau = \frac{(k-1)!}{2 t^k} .$$

It follows that

$$\frac{K_k(\sqrt{s}\, r)}{K_k(\sqrt{s})} \leq \frac{2^{k+1}}{r^k} .$$

This is also valid for $k = 0$.

In particular, the above two bounds show that the series for $\alpha(r, \theta)$ converges absolutely for $r > 3$ and,

$$
\begin{aligned}
|\alpha(r, \theta)| &\leq C\, e^{-\sqrt{s}(r-1)/2} \sum_{k=0}^\infty \frac{2^{k/2}}{r^{k/2}} \left[|a_k| + |b_k| \right] \\
&\leq C\, e^{-\sqrt{s}(r-1)/2} \frac{r^2}{r^2 - 4} \left\{ \sum_{k=0}^\infty [a_k^2 + b_k^2] \right\}^{1/2} \\
&\leq C\, e^{-\sqrt{s}\, r/4} \|\alpha\|_{L^2(\mathbb{S}^1)} \text{ for } r > 3
\end{aligned}
\tag{108}
$$

as announced. □

8 Interior estimates for the boundary value problem

In this section, we detail regularity estimates for the solutions $\tilde{W}_{\varepsilon,1}$ and $\tilde{W}_{\varepsilon,2}$ of the boundary value problems (30) which appear in the proof of Theorem 2.

8.1 Regularity of $\tilde{W}_{\varepsilon,1}$

Our result is the following

Proposition 9 *Let* $\Omega_\tau^\varepsilon = \omega_\tau^\varepsilon \times (-L, L)$. *For* $\kappa > 0$, $\tau > \kappa\, \varepsilon^{\frac{1-\eta}{2(1+\eta)}}$ *with* $\eta \in \left(\frac{1}{2}, 1\right)$, *and* $0 < \nu < 2\left(\eta - \frac{1}{2}\right)$, *the solution* $W_{\varepsilon,1}$ *of* (35) *satisfies*

$$
\begin{aligned}
\left\| \tilde{W}_{\varepsilon,1} - \tilde{V}_1 \right\|_{L^\infty(\Omega_\tau^\varepsilon)} + \left\| \nabla \tilde{W}_{\varepsilon,1} - \nabla \tilde{V}_1 \right\|_{L^\infty(\Omega_\tau^\varepsilon)} &\leq C(\kappa, \eta) \|\Phi_b\|_{C^1(\bar{\Omega})}, \\
\left[\nabla \tilde{W}_{\varepsilon,1} - \nabla \tilde{V}_1 \right]_{C^\nu(\Omega_\tau^\varepsilon)} &\leq C(\kappa, \eta, \nu) \|\Phi_b\|_{C^1(\bar{\Omega})}.
\end{aligned}
$$

Proof We apply a Fourier series decomposition in x_3. We write

$$\tilde{W}_{\varepsilon,1}(x', x_3) \sim \sum_{n=1}^\infty \tilde{w}_{\varepsilon,1,n}(x') \sin\left(\frac{n\pi}{2} \left(\frac{x_3}{L} + 1\right)\right),$$

$$\tilde{V}_1(x', x_3) \sim \sum_{n=1}^\infty \tilde{v}_{1,n}(x') \sin\left(\frac{n\pi}{2} \left(\frac{x_3}{L} + 1\right)\right).$$

As $\tilde{W}_{\varepsilon,1} - \tilde{V}_1$ is harmonic therefore smooth in $\Omega \setminus Q_\varepsilon$, the Fourier expansions of $\tilde{W}_{\varepsilon,1} - \tilde{V}_1$ and $\nabla(\tilde{W}_{\varepsilon,1} - \tilde{V}_1)$ converge pointwise to themselves in Ω_τ^ε.

Now $\tilde{w}_{\varepsilon,1,n} \in H_0^1(\omega)$ satisfies

$$\operatorname{div}_2(a_\varepsilon \nabla_2 \tilde{w}_{\varepsilon,1,n}) + \frac{n^2 \pi^2}{4L^2} a_\varepsilon \, \tilde{w}_{\varepsilon,1,n} = \operatorname{div}_2(\mathbf{h}_n)$$

where $\mathbf{h}_n = (h_n^1, h_n^2)$ is given by

$$h_n^i = \int_{-L}^{L} \partial_i \phi_1 \sin\left(\frac{n\pi}{2}\left(\frac{x_3}{L}+1\right)\right).$$

Since $\partial_{33}\phi_1 = 0$, we see from an integration by parts that

$$\|h_n^i\|_{L^\infty(\omega)} \leq \frac{C}{n}\|\nabla\phi_1\|_{L^\infty(\Omega)} \leq \frac{C}{n}\|\Phi_b\|_{C^1(\bar{\Omega})}. \tag{109}$$

Lemma 3, applied with $\tilde{p} = \infty$, shows that for any $\eta < 1$,

$$\|\tilde{w}_{\varepsilon,1,n}\|_{L^\infty(\bar{\Omega})} \leq \frac{C}{n^\eta}\|h_n^i\|_{L^\infty(\omega)}, \tag{110}$$

Proposition 2 together with (109) and (110) now shows that, for every $0 < \nu < \eta < 1$,

$$\|\tilde{w}_{\varepsilon,1,n} - \tilde{v}_{1,n}\|_{L^\infty(\omega_\tau^\varepsilon)} + \|\nabla_2 \tilde{w}_{\varepsilon,1,n} - \nabla_2 \tilde{v}_{1,n}\|_{L^\infty(\omega_\tau^\varepsilon)} \leq \frac{C}{n^{2\eta+1}}\|\Phi_b\|_{C^1(\bar{\Omega})},$$

and

$$[\nabla_2 \tilde{w}_{\varepsilon,1,n} - \nabla_2 v]_{C^\nu(\omega_\tau^\varepsilon)} \leq \frac{C}{n^{2\eta+1-\nu}}\|\Phi_b\|_{C^1(\bar{\Omega})},$$

Following a variant of the proof of Theorem 1, we obtain consecutively the following estimates using the indicated choice of η,

$$\left\|\tilde{W}_{\varepsilon,1} - \tilde{V}_1\right\|_{L^\infty(\Omega_\tau^\varepsilon)} + \left\|\nabla_2 \tilde{W}_{\varepsilon,1} - \nabla_2 \tilde{V}_1\right\|_{L^\infty(\Omega_\tau^\varepsilon)} \leq C\|\Phi_b\|_{C^1(\bar{\Omega})} \qquad \text{with any } \eta > 0,$$

$$[\nabla_2 \tilde{W}_{\varepsilon,1} - \nabla_2 \tilde{V}_1]_{C^\nu(\Omega_\tau^\varepsilon)} \leq C\|\Phi_b\|_{C^1(\bar{\Omega})} \qquad \text{with any } \eta > \frac{\nu}{2},$$

$$\left\|\partial_{x_3}\tilde{W}_{\varepsilon,1} - \partial_{x_3}\tilde{V}_1\right\|_{L^\infty(\Omega_\tau^\varepsilon)} \leq C\|\Phi_b\|_{C^1(\bar{\Omega})} \qquad \text{with any } \eta > \frac{1}{2},$$

$$[\partial_{x_3}\tilde{W}_{\varepsilon,1} - \partial_{x_3}\tilde{V}_1]_{C^\nu(\Omega_\tau^\varepsilon)} \leq C\|\Phi_b\|_{C^1(\bar{\Omega})} \qquad \text{with any } \eta > \frac{1}{2} + \frac{\nu}{2}.$$

This completes the proof. $\square$

8.2 Regularity of $\tilde{W}_{\varepsilon,2}$

Our result is the following

Proposition 10 *Let* $\Omega_\tau^\varepsilon = \omega_\tau^\varepsilon \times (-L, L)$. *For* $\kappa > 0$, $\tau > \kappa \, \varepsilon^{\frac{1-\eta}{2(1+\eta)}}$ *with* $\eta \in \left(\frac{2}{3}, 1\right)$, *and* $0 < \nu < 3\left(\eta - \frac{2}{3}\right)$, *the solutions* $\tilde{W}_{\varepsilon,2}$ *and* $\tilde{V}_{\varepsilon,2}$ *of (38) and (43), respectively, satisfy*

$$\left\|\tilde{W}_{\varepsilon,2} - \tilde{V}_{\varepsilon,2}\right\|_{L^\infty(\Omega_\tau^\varepsilon)} + \left\|\nabla\tilde{W}_{\varepsilon,2} - \nabla\tilde{V}_{\varepsilon,2}\right\|_{L^\infty(\Omega_\tau^\varepsilon)} \leq C(\kappa, \eta)\varepsilon^\sigma\|\Phi_b\|_{C^1(\bar{\Omega})},$$

$$\left[\nabla\tilde{W}_{\varepsilon,2} - \nabla\tilde{V}_{\varepsilon,2}\right]_{C^\nu(\Omega_\tau^\varepsilon)} \leq C(\kappa, \eta, \nu)\varepsilon^\sigma\|\Phi_b\|_{C^1(\bar{\Omega})},$$

Proof As before, we decompose $\tilde{W}_{\varepsilon,2}$ in a Fourier series along the third direction, i.e.

$$\tilde{W}_{\varepsilon,2} \sim \sum_{n=1}^{\infty} \tilde{w}_{\varepsilon,2,n} \sin\left(\frac{n\pi}{2L}(x_3+L)\right),$$

$$\tilde{V}_{\varepsilon,2} \sim \sum_{n=1}^{\infty} \tilde{v}_{2,n} \sin\left(\frac{n\pi}{2L}(x_3+L)\right).$$

As usual, $\tilde{W}_{\varepsilon,2} - \tilde{V}_{\varepsilon,2}$ is harmonic and regular in $\Omega \setminus Q_\varepsilon$. Thus, we can sum the estimates on the Fourier coefficients to obtain estimates on $\tilde{W}_{\varepsilon,2} - \tilde{V}_{\varepsilon,2}$ and $\nabla(\tilde{W}_{\varepsilon,2} - \tilde{V}_{\varepsilon,2})$ in Ω_τ^ε.

The problem satisfied by $\tilde{w}_{\varepsilon,2,n}$ is now, in $H_0^1(\omega)$,

$$-\text{div}_2\left(a_\varepsilon \nabla_2 \tilde{w}_{\varepsilon,2,n}\right) + \lambda a_\varepsilon \tilde{w}_{\varepsilon,2,n} = a_\varepsilon A_{\varepsilon,n} + a_\varepsilon B_{\varepsilon,n} + C_{\varepsilon,n},$$

where $\lambda = \frac{n^2\pi^2}{4L^2}$, and

$$A_{\varepsilon,n} = \int_{-L}^{L} \sin\left(\sqrt{\lambda}(t+L)\right) \partial_3\zeta(\cdot,t)\partial_3 W_{\varepsilon,2}(\cdot,t)dt,$$

$$B_{\varepsilon,n} = \int_{-L}^{L} \sin\left(\sqrt{\lambda}(t+L)\right) \partial_{33}\zeta(\cdot,t)W_{\varepsilon,2}(\cdot,t)dt,$$

$$C_{\varepsilon,n} = \int_{-L}^{L} \sin\left(\sqrt{\lambda}(t+L)\right) g_\varepsilon(\cdot,t)dt.$$

Proposition 2 together with Proposition 11, now shows that for all $0 < \nu < \eta < 1$,

$$\|\tilde{w}_{\varepsilon,2,n} - \tilde{v}_{2,n}\|_{L^\infty(\omega_\tau^\varepsilon)} + \|\nabla_2 \tilde{w}_{\varepsilon,2,n} - \nabla_2 \tilde{v}_{2,n}\|_{L^\infty(\omega_\tau^\varepsilon)} \le \varepsilon^\sigma \frac{C}{n^{3\eta}} \|\Phi_b\|_{C^1(\bar{\Omega})},$$

and

$$[\nabla_2 \tilde{w}_{\varepsilon,2,n} - \nabla_2 \tilde{v}_{2,n}]_{C^\nu(\omega_\tau^\varepsilon)} \le \varepsilon^\sigma \frac{C}{n^{3\eta-\nu}} \|\Phi_b\|_{C^1(\bar{\Omega})}.$$

Following a variant of the proof of Theorem 1, we obtain consecutively the following estimates using the indicated choice of η,

$$\left\|\tilde{W}_{\varepsilon,2} - \tilde{V}_2\right\|_{L^\infty(\Omega_\tau^\varepsilon)} + \left\|\nabla_2 \tilde{W}_{\varepsilon,2} - \nabla_2 \tilde{V}_2\right\|_{L^\infty(\Omega_\tau^\varepsilon)} \le C\,\varepsilon^\sigma \|\Phi_b\|_{C^1(\bar{\Omega})} \qquad \text{with any } \eta > \frac{1}{3},$$

$$[\nabla_2 \tilde{W}_{\varepsilon,2} - \nabla_2 \tilde{V}_2]_{C^\nu(\Omega_\tau^\varepsilon)} \le C\,\varepsilon^\sigma \|\Phi_b\|_{C^1(\bar{\Omega})} \qquad \text{with any } \eta > \frac{1}{3} + \frac{\nu}{3},$$

$$\left\|\partial_{x_3} \tilde{W}_{\varepsilon,2} - \partial_{x_3} \tilde{V}_2\right\|_{L^\infty(\Omega_\tau^\varepsilon)} \le C\varepsilon^\sigma \|\Phi_b\|_{C^1(\bar{\Omega})} \qquad \text{with any } \eta > \frac{2}{3},$$

$$[\partial_{x_3}\tilde{W}_{\varepsilon,2} - \partial_{x_3}\tilde{V}_2]_{C^\nu(\Omega_\tau^\varepsilon)} \le C\varepsilon^\sigma \|\Phi_b\|_{C^1(\bar{\Omega})} \qquad \text{with any } \eta > \frac{2}{3} + \frac{\nu}{3},$$

which concludes our argument. □

Proposition 11 *We have the following estimate*

$$\|\sqrt{a_\varepsilon}A_{\varepsilon,n}\|_{L^2(\omega)} + \|\sqrt{a_\varepsilon}B_{\varepsilon,n}\|_{L^2(\omega)} + \|\sqrt{a_\varepsilon}C_{\varepsilon,n}\|_{L^2(\omega)} \le \frac{C}{\sqrt{\lambda}}\varepsilon^{\sigma+2}\|\Phi\|_{C^1(\bar{\Omega})}.$$

As a consequence,

$$\|\tilde{w}_{\varepsilon,2,n}\|_{L^\infty(\omega)} \le C(\eta)\frac{\varepsilon^\sigma}{\lambda^\eta}\|\Phi\|_{C^1(\bar{\Omega})}.$$

for any $\eta < 1$.

Proof Note that (42) shows that $C_{\varepsilon,n}$ satisfies

$$\|C_{\varepsilon,n}\|_{L^\infty(\omega)} \leq \frac{C}{\sqrt{\lambda}} \|\phi_2\|_{L^\infty(\Omega)} .$$

Turning to $B_{\varepsilon,n}$ an integration by parts shows that

$$|B_{\varepsilon,n}| \leq \frac{C}{\sqrt{\lambda}} \left(\|W_{\varepsilon,2}\|_{L^\infty(\Omega)} + \int_{-L}^{L} \mathbf{1}_{\omega_3 \times (-l',l')} |\partial_3 W_{\varepsilon,2}(\cdot,t)| dt \right),$$

which in turn shows that

$$\int_\omega a_\varepsilon |B_{\varepsilon,n}|^2 dx' \leq \frac{C}{\lambda} \left(\|W_{\varepsilon,2}\|_{L^\infty(\Omega)} + \int_{\omega_3 \times (-l',l')} a_\varepsilon |\partial_3 W_{\varepsilon,2}|^2 dx \right).$$

Thanks to (39) and the interior estimate (41), we have obtained that

$$\|\sqrt{a_\varepsilon} B_{\varepsilon,n}\|_{L^2(\omega)} \leq \frac{C}{\sqrt{\lambda}} \|\phi_2\|_{L^\infty(\Omega)} .$$

We proceed with $A_{\varepsilon,n}$ in a similar way. After an integration by parts, we derive that

$$|A_{\varepsilon,n}|^2 \leq \frac{C}{\lambda} \left(\int_{-L}^{L} \mathbf{1}_{\omega_3 \times (-l',l')} \left(|\partial_3 W_{\varepsilon,2}(\cdot,t)|^2 + |\partial_{33} W_{\varepsilon,2}(\cdot,t)|^2 \right) dt \right)$$

Multiplying this quantity by a_ε, and integrating we obtain thanks to (39) and (41)

$$\|\sqrt{a_\varepsilon} A_{\varepsilon,n}\|_{L^2(\omega)} \leq \frac{C}{\sqrt{\lambda}} \|\phi_2\|_{L^\infty(\Omega)} .$$

We have obtained that

$$\|\sqrt{a_\varepsilon} A_{\varepsilon,n}\|_{L^2(\omega)} + \|\sqrt{a_\varepsilon} B_{\varepsilon,n}\|_{L^2(\omega)} + \|C_{\varepsilon,n}\|_{L^2(\omega)} \leq \frac{C}{\sqrt{\lambda}} \|\phi_2\|_{L^\infty(\Omega)} .$$

The first assertion now follows from Proposition 5.

Finally, using Lemma 1, we know that for any $1 < \alpha < 2$, and any $\beta < 1 - \alpha/2$,

$$\|\tilde{w}_{\varepsilon,2,n}\|_{L^\infty(\omega)} \leq \frac{C(\alpha,\beta)}{\varepsilon^\alpha \lambda^\beta} \left(\|\sqrt{a_\varepsilon} A_{\varepsilon,n}\|_{L^2(\omega)} + \|\sqrt{a_\varepsilon} B_{\varepsilon,n}\|_{L^2(\omega)} + \|C_{\varepsilon,n}\|_{L^2(\omega)} \right)$$

$$\leq C(\alpha,\beta) \frac{\varepsilon^{\sigma+2-\alpha}}{\lambda^{1/2+\beta}} \|\Phi\|_{C^1(\Omega)}$$

$$\leq C(\beta) \varepsilon^\sigma \lambda^{-1/2-\beta} \|\Phi\|_{C^1(\Omega)} .$$

$\square$

A Proofs of technical lemmas and propositions

A.1 Proof of Proposition 1

Note that we have

$$
\begin{cases}
-\Delta W_* + \gamma\,(W_* - V_*) = 0 \text{ in } \Omega_0, \\
-\kappa\,\partial_{33}^2 V_* + \gamma\,(V_* - W_*) = 0 \text{ in } \Omega_0.
\end{cases}
$$

And a simple induction shows that W_* and V_* are infinitely differentiable with respect to x_3 in Ω_0. To gain analyticity, we resort to estimates. Fix a ball $B(p^0, R)$ in Ω_0. We first derive an integral representation for $W_*(p^0)$. For any $r > 0$, the function

$$
G_{p^0,r}(x) := \frac{\cosh(\sqrt{\gamma}|x - p^0|)}{4\pi|x - p^0|} - \coth(\sqrt{\gamma}r)\,\frac{\sinh(\sqrt{\gamma}|x - p^0|)}{4\pi|x - p^0|}
$$

satisfies

$$
\begin{cases}
(-\Delta + \gamma)G_{p^0,r} = \delta_{p^0} & \text{in } B(p^0, r), \\
G_{p^0,r} = 0 & \text{on } \partial B(p^0, r).
\end{cases}
$$

A direct application of Green's formula then yields that for $0 < r < R$, there holds

$$
W_*(p^0) = \gamma \int_{B(p^0,r)} G_{p^0,r}(x)\,V_*(x)\,dx + \frac{\sqrt{\gamma}}{4\pi\,r\,\sinh(\sqrt{\gamma}r)} \int_{\partial B(p^0,r)} W_*(x)\,d\sigma(x).
$$

Here we have used

$$
\frac{\partial}{\partial n}G_{p^0,r}\Big|_{\partial B(p^0,r)} = -\frac{\sqrt{\gamma}}{4\pi\,r\,\sinh(\sqrt{\gamma}r)}.
$$

It follows that

$$
W_*(p^0) \int_0^R r\,\sinh(\sqrt{\gamma}r)\,dr
$$
$$
= \gamma \int_0^R r\,\sinh(\sqrt{\gamma}r)\,dr \int_{B(p^0,r)} G_{p^0,r}(x)\,V_*(x)\,dx + \frac{\sqrt{\gamma}}{4\pi} \int_{B(p^0,R)} W_*(x)\,dx.
$$

Applying this identity to $\partial_{x_3}^{(n+1)} W_*$ we obtain

$$
\begin{aligned}
\partial_{x_3}^{(n+1)} W_*(p^0) \int_0^R r\,\sinh(\sqrt{\gamma}r)\,dr \quad &= \gamma \int_0^R r\,\sinh(\sqrt{\gamma}r)\,dr \int_{B(p^0,r)} G_{p^0,r}(x)\,\partial_{x_3}^{(n+1)} V_*(x)\,dx \\
&\quad + \frac{\sqrt{\gamma}}{4\pi} \int_{B(p^0,R)} \partial_{x_3}^{(n+1)} W_*(x)\,dx \\
&= \gamma \int_0^R r\,\sinh(\sqrt{\gamma}r)\,dr \int_{B(p^0,r)} G_{p^0,r}(x)\,\partial_{x_3}^{(n+1)} V_*(x)\,dx \\
&\quad + \frac{\sqrt{\gamma}}{4\pi} \int_{\partial B(p^0,R)} \partial_{x_3}^{(n)} W_*(x)\,\frac{x_3 - p_3^0}{|x - p_0|}\,d\sigma(x).
\end{aligned}
$$

Where we used the notation $p^0 = (p_1^0, p_2^0, p_3^0)$. Noting that $0 \leq G_{p^0,r}(x) \leq \frac{\cosh(\sqrt{\gamma}|x-p^0|)}{|x-p^0|}$, we deduce that

$$\left|\partial_{x_3}^{(n+1)} W_*(p^0)\right| \int_0^R r \sinh(\sqrt{\gamma} r)\, dr$$
$$\leq C \Big[\int_0^R r \sinh(\sqrt{\gamma} r)\, dr \int_0^r s \cosh(\sqrt{\gamma} s)\, ds \|\partial_{x_3}^{(n+1)} V_*\|_{L^\infty(B(p^0,R))}$$
$$+ R^2 \|\partial_{x_3}^{(n)} W_*\|_{L^\infty(B(p_0,R))} \Big],$$

and we have bounded $\partial_{x_3}^{(n+1)} W_*(p^0)$ in terms of lower derivatives by

$$\left|\partial_{x_3}^{(n+1)} W_*(p^0)\right| \leq C \Big[R^2 \|\partial_{x_3}^{(n+1)} V_*\|_{L^\infty(B(p^0,R))} + \frac{1}{R} \|\partial_{x_3}^{(n)} W_*\|_{L^\infty(B(p_0,R))} \Big]. \tag{111}$$

Let us now turn to V_*. For $x_3 \in (p_3^0 - R, p_3^0 + R)$ we have

$$\partial_{x_3}^{(n)} V_*(p_1^0, p_2^0, x_3) = \alpha_+ \exp\left(\sqrt{\frac{\gamma}{\kappa}} x_3\right) + \alpha_- \exp\left(-\sqrt{\frac{\gamma}{\kappa}} x_3\right)$$
$$+ \sqrt{\frac{\gamma}{\kappa}} \int_{p_3^0-R}^{x_3} \sinh\left(\sqrt{\frac{\gamma}{\kappa}}(s - x_3)\right) \partial_{x_3}^{(n)} W_*(p_1^0, p_2^0, s)\, ds$$

where the constants $\alpha_\pm$ satisfy

$$|\alpha_\pm| \leq C \Big[\frac{1}{R} \|\partial_{x_3}^{(n)} V_*\|_{L^\infty(B(p^0,R))} + R \|\partial_{x_3}^{(n)} W_*\|_{L^\infty(B(p^0,R))} \Big].$$

In particular,

$$\partial_{x_3}^{(n+1)} V_*(p^0) = \alpha_+ \sqrt{\frac{\gamma}{\kappa}} \exp\left(\sqrt{\frac{\gamma}{\kappa}} p_3^0\right) - \alpha_- \sqrt{\frac{\gamma}{\kappa}} \exp\left(-\sqrt{\frac{\gamma}{\kappa}} p_3^0\right)$$
$$- \frac{\gamma}{\kappa} \int_{p_3^0-R}^{p_3^0} \cosh\left(\sqrt{\frac{\gamma}{\kappa}}(s - p_3^0)\right) \partial_{x_3}^{(n)} W_*(p_1^0, p_2^0, s)\, ds$$

which provides a bound of $\partial_{x_3}^{(n+1)} V_*(p^0)$ in terms of lower derivatives given by

$$\left|\partial_{x_3}^{(n+1)} V_*(p^0)\right| \leq C \Big[R \|\partial_{x_3}^{(n)} W_*\|_{L^\infty(B(p^0,R))} + \frac{1}{R} \|\partial_{x_3}^{(n)} V_*\|_{L^\infty(B(p_0,R))} \Big]. \tag{112}$$

Using (112) for the first term on the right hand side of (111), we see for all R sufficiently small that

$$\left|\partial_{x_3}^{(n+1)} W_*(p^0)\right| \leq C \Big[R \|\partial_{x_3}^{(n)} V_*\|_{L^\infty(B(p^0,R))} + \frac{1}{R} \|\partial_{x_3}^{(n)} W_*\|_{L^\infty(B(p_0,R))} \Big].$$

This inequality together with (112) shows that

$$\left|\partial_{x_3}^{(n+1)} W_*(p^0)\right| + \left|\partial_{x_3}^{(n+1)} V_*(p^0)\right| \leq \frac{C_0}{R} \Big[\|\partial_{x_3}^{(n)} W_*\|_{L^\infty(B(p^0,R))} + \|\partial_{x_3}^{(n)} V_*\|_{L^\infty(B(p_0,R))} \Big]. \tag{113}$$

To conclude, we will show by induction that (113) implies

$$\left|\partial_{x_3}^{(n)} W_*(p^0)\right| + \left|\partial_{x_3}^{(n)} V_*(p^0)\right| \leq \frac{C_0^n n^n}{R^n} \Big[\|W_*\|_{L^\infty(B(p^0,R))} + \|V_*\|_{L^\infty(B(p_0,R))} \Big]. \tag{114}$$

Since this holds for any value of n, analyticity of W_* and V_* in the x_3 direction then follows from Sterling's formula.

By (113), the above estimate holds for $n = 1$. Assume that it holds for some n. Using estimate (113) we find,

$$\left|\partial_{x_3}^{(n+1)} W_*(p^0)\right| + \left|\partial_{x_3}^{(n+1)} V_*(p^0)\right| \leq \frac{C_0(n+1)}{R}\left[\|\partial_{x_3}^{(n)} W_*\|_{L^\infty(B(p^0, \frac{R}{n+1}))} + \|\partial_{x_3}^{(n)} V_*\|_{L^\infty(B(p_0, \frac{R}{n+1}))}\right].$$

Note that every point in $B(p_0, \frac{R}{n+1})$ is contained in a ball of radius $\frac{nR}{n+1}$ which is contained in $B(p^0, R)$. Thus, applying (114) to the right hand side we obtain

$$\left|\partial_{x_3}^{(n+1)} W_*(p^0)\right| + \left|\partial_{x_3}^{(n+1)} V_*(p^0)\right|$$

$$\leq \frac{C_0(n+1)}{R}\frac{C_0^n\, n^n}{(\frac{nR}{n+1})^n}\left[\|W_*\|_{L^\infty(B(p^0, R))} + \|V_*\|_{L^\infty(B(p_0, R))}\right]$$

$$= \frac{C_0^{n+1}(n+1)^{n+1}}{R^{n+1}}\left[\|W_*\|_{L^\infty(B(p^0, R))} + \|V_*\|_{L^\infty(B(p_0, R))}\right].$$

which is our induction thesis.

A.2 Proof of Proposition 5

Most properties are easily verified by inspection. Regarding ϕ_1 let us check its regularity in the support of $\nabla c_\varepsilon^\sigma \times [-L, L]$. The definition of ϕ_1 being local, it suffices to look at one cell, centered in $(0,0)$. We have

$$\partial_3 \phi_1 = \partial_3 \phi_L\, c_\varepsilon^\sigma + (1 - c_\varepsilon^\sigma)\, \partial_3 \phi_L\, (0, 0, x_3),$$

and for $i = 1, 2$,

$$|\partial_i \phi_1| = \left|\partial_i \phi_L\, c_\varepsilon^\sigma + \frac{1}{\varepsilon^2 \ln\left(\varepsilon^\sigma/(2\varepsilon\, r_\varepsilon)\right)}\frac{x_i}{x_1^2 + x_2^2}\left(x_1 \partial_1 \phi_L\,(\zeta, x_3) + x_2 \partial_2 \phi_L\,(\zeta, x_3)\right)\right|$$

$$\leq C\left(\|\partial_1 \phi_L\|_\infty + \|\partial_2 \phi_L\|_\infty\right),$$

since, thanks to (3),

$$\varepsilon^2 \ln\left(\varepsilon^{\sigma-1}/r_\varepsilon\right) \leq C.$$

The conclusion for ϕ_1 follows.

Regarding ϕ_2, note that

$$\phi_2(x) = \sum_{(m,n)\in I_\varepsilon} (1 - c_\varepsilon^\sigma)\, \mathbf{1}_{m,n,\varepsilon}\,(x)\left(\phi_L\,(m\varepsilon, n\varepsilon, x_3) - \phi_L(x)\right).$$

Again, the definition of ϕ_2 is local. In the cell $[-\varepsilon/2, \varepsilon/2]^3$, we have

$$|\phi_2| \leq Cr\, \|\nabla\phi_L\|_{L^\infty(\Omega)} \text{ for } r \leq \varepsilon r_\varepsilon,$$

$$|\phi_2| \leq Cr\frac{\ln(\varepsilon^\sigma/2) - \ln r}{\ln(\varepsilon^\sigma/2) - \ln(\varepsilon r_\varepsilon)}\, \|\nabla\phi_L\|_{L^\infty(\Omega)} \text{ for } \varepsilon r_\varepsilon \leq r \leq \varepsilon^\sigma/2,$$

$$\phi_2 = 0 \text{ for } \varepsilon^\sigma \leq 2r,$$

with $r = \sqrt{x_1^2 + x_2^2}$. Thanks to (3), this yields the global bound

$$|\phi_2| \leq C\varepsilon^{\sigma+2}\, \|\nabla\phi_L\|_{L^\infty(\Omega)}.$$

A.3 Proof of Lemma 9

Without loss of generality, we can assume that the centre of $D_{m,n,\varepsilon}$ is the origin.

Note that from (88) we know that

$$\|\tilde{u}_\varepsilon\|_{L^\infty(\omega\setminus D_\varepsilon)} \le \sup_{m,n}|u_{m,n}| =: M \;.$$

Remember that two linearly independent radial solutions of

$$-\Delta u + \lambda u = 0$$

for $|x| > 0$ are $I_0\left(\sqrt{\lambda}\cdot\right)$ and $K_0\left(\sqrt{\lambda}\cdot\right)$, the modified Bessel functions of the first and second kind. They are defined by

$$K_0(r) = \int_0^\infty e^{-r\cosh t}\,dt \;, \tag{115}$$

$$I_0(r) = \sum_{n=0}^\infty \frac{\left(r^2/4\right)^n}{(n!)^2} \;, \tag{116}$$

see, e.g. [32]. Introduce

$$\psi(x) = \psi(|x|) = \alpha\,I_0(\sqrt{\lambda}|x|) + \beta\,K_0(\sqrt{\lambda}|x|),$$

where α and β are chosen so that

$$\psi(\varepsilon\,r_\varepsilon) = u_{m,n}, \qquad \psi(\varepsilon/2) = M.$$

Thanks to the maximum principle, $\tilde{u}_\varepsilon(x) \le \psi(x)$. Since the two functions agree on $\partial D_{m,n,\varepsilon}$, the normal derivative of ψ at $|x| = \varepsilon r_\varepsilon$ gives an upper bound for the normal derivative of $\tilde{u}_\varepsilon$ at $\partial D_{m,n,\varepsilon}$:

$$\sup_{\partial D_{m,n,\varepsilon}} \frac{\partial\tilde{u}_\varepsilon}{\partial n} \le |\psi'(\varepsilon\,r_\varepsilon)|. \tag{117}$$

We thus proceed to estimate $|\psi'(\varepsilon\,r_\varepsilon)|$. The constants α and β are given by

$$\alpha = \frac{u_{m,n}K_0\left(\sqrt{\lambda}\varepsilon/2\right) - MK_0\left(\sqrt{\lambda}r_\varepsilon\varepsilon\right)}{I_0\left(\sqrt{\lambda}r_\varepsilon\varepsilon\right)K_0\left(\sqrt{\lambda}\varepsilon/2\right) - K_0\left(\sqrt{\lambda}r_\varepsilon\varepsilon\right)I_0\left(\sqrt{\lambda}\varepsilon/2\right)},$$

$$\beta = \frac{-u_{m,n}I_0\left(\sqrt{\lambda}\varepsilon/2\right) + MI_0\left(\sqrt{\lambda}r_\varepsilon\varepsilon\right)}{I_0\left(\sqrt{\lambda}r_\varepsilon\varepsilon\right)K_0\left(\sqrt{\lambda}\varepsilon/2\right) - K_0\left(\sqrt{\lambda}r_\varepsilon\varepsilon\right)I_0\left(\sqrt{\lambda}\varepsilon/2\right)}.$$

The proof relies on precise estimates on α and β for various regime of λ.

It is convenient to introduce the notation $z = \sqrt{\lambda}\varepsilon$. Let us first consider the case when

$$z \le r_\varepsilon^{-\frac{1}{3}},$$

We shall use an *ad-hoc* bound, easily verifiable using the Frobenius decomposition of K_0, see e.g. [32]. For all $x > 0$, we have

$$K_0(x) = I_0(x)\left(-\ln\left(\frac{x}{2}\right) - \gamma + \ln(x+e)\frac{t(x)}{1+t(x)}R_0(x)\right), \tag{118}$$

where $t(x) = \frac{x^2}{4+x^{5/2}\sqrt{2\pi}}e^x$ and R_0 satisfies $\frac{2}{5} \le R_0(x) \le \frac{5}{4}$ for all $x > 0$. We now compute

$$\frac{I_0\left(r_\varepsilon z\right) K_0\left(\frac{1}{2}z\right) - K_0\left(r_\varepsilon z\right) I_0\left(\frac{1}{2}z\right)}{I_0\left(\frac{1}{2}z\right) I_0\left(r_\varepsilon z\right)}$$
$$= \ln\left(2r_\varepsilon\right) + \ln\left(\frac{1}{2}z + e\right)\frac{t\left(\frac{1}{2}z\right)}{1 + t\left(\frac{1}{2}z\right)}R_0\left(\frac{1}{2}z\right) - \ln\left(r_\varepsilon z + e\right)\frac{t\left(r_\varepsilon z\right)}{1 + t\left(r_\varepsilon z\right)}R_0\left(r_\varepsilon z\right).$$

Note that

$$\left|\ln(x/2 + e)\frac{t(x/2)}{1 + t(x/2)}R_0(x/2)\right| \le |\max(1, \ln(x))|$$
$$\le \frac{1}{3}|\ln(r_\varepsilon)| \text{ when } x < \frac{1}{r_\varepsilon^{1/3}}.$$

Therefore we have

$$I_0\left(r_\varepsilon z\right) K_0\left(\frac{1}{2}z\right) - K_0\left(r_\varepsilon z\right) I_0\left(\frac{1}{2}z\right) = I_0\left(\frac{1}{2}z\right) I_0\left(r_\varepsilon z\right)\ln(r_\varepsilon)\left(1 + E_\varepsilon\right), \tag{119}$$

with

$$|E_\varepsilon| \le \frac{2}{3} \text{ for all } z \le r_\varepsilon^{-1/3}.$$

We can now estimate α and β as follows

$$|\alpha| \le \left|\frac{u_{m,n}K_0\left(z/2\right)}{I_0\left(r_\varepsilon z\right) K_0\left(\frac{1}{2}z\right) - K_0\left(r_\varepsilon z\right) I_0\left(\frac{1}{2}z\right)}\right|$$
$$+ \left|M\frac{K_0\left(r_\varepsilon z\right)}{I_0\left(r_\varepsilon z\right) K_0\left(\frac{1}{2}z\right) - K_0\left(r_\varepsilon z\right) I_0\left(\frac{1}{2}z\right)}\right|$$
$$\le M\frac{\left|-\ln\left(\frac{z}{2}\right) - \gamma + \ln(\frac{z}{2} + e)\frac{t(z/2)}{1+t(z/2)}R_0(z/2)\right|}{I_0\left(r_\varepsilon z\right)|\ln(r_\varepsilon)|\left(1 + E_\varepsilon\right)}$$
$$+ M\frac{\left|-\ln\left(r_\varepsilon z\right) - \gamma + \ln(r_\varepsilon z + e)\frac{t(zr_\varepsilon)}{1+t(zr_\varepsilon)}R_0(r_\varepsilon z)\right|}{I_0\left(r_\varepsilon z\right)|\ln(r_\varepsilon)|\left(1 + E_\varepsilon\right)}$$
$$\le 3\frac{M}{I_0\left(r_\varepsilon z\right)} + 9\frac{M}{I_0\left(\frac{1}{2}z\right)} \text{ when } z \le r_\varepsilon^{-1/3}.$$

Therefore

$$|\alpha| \le C\frac{M}{I_0\left(r_\varepsilon z\right)} \le CM.$$

Similarly, we obtain

$$|\beta| \le \frac{M\left|I_0\left(z/2\right)\right| + M\left|I_0\left(r_\varepsilon z\right)\right|}{\left|I_0\left(r_\varepsilon z\right) K_0\left(\frac{1}{2}z\right) - K_0\left(r_\varepsilon z\right) I_0\left(\frac{1}{2}z\right)\right|} \le C\frac{M}{I_0\left(r_\varepsilon z\right)|\ln(r_\varepsilon)|} \le C\frac{M}{|\ln(r_\varepsilon)|}.$$

We are now in position to bound ψ', namely

$$|\psi'(t)| \le \sqrt{\lambda}|\alpha|I_1\left(\sqrt{\lambda}t\right) + \sqrt{\lambda}|\beta|K_1\left(\sqrt{\lambda}t\right).$$

At $t = \varepsilon r_\varepsilon$, since $\sqrt{\lambda}\varepsilon r_\varepsilon \le r_\varepsilon^{2/3} < 1$, we deduce that $I_1\left(\sqrt{\lambda}t\right) \le r_\varepsilon^{2/3}$. Using the fact that for all $x > 0$, $xK_1(x) \le 1$, we obtain

$$\le \sqrt{\lambda}CM\left(r_\varepsilon^{2/3} + \frac{1}{\sqrt{\lambda}\varepsilon r_\varepsilon|\ln(r_\varepsilon)|}\right) \le CM\frac{1}{\varepsilon r_\varepsilon|\ln(r_\varepsilon)|}.$$

We have thus shown that

$$|\psi'(\varepsilon r_\varepsilon)| \le C\theta(\lambda,\varepsilon)\,M \text{ for } z < r_\varepsilon^{-\frac{1}{3}}. \tag{120}$$

Let us now turn to the case when $z \ge r_\varepsilon^{-1/3}$. In this case, we easily show that

$$\alpha \approx M\frac{1}{I_0\left(\frac{1}{2}z\right)} \text{ and } \beta \approx u_{m,n}\frac{1}{K_0\left(r_\varepsilon z\right)},$$

leading to the bounds

$$|\psi'(\varepsilon r_\varepsilon)| \le \sqrt{\lambda}CM\left(\frac{I_1\left(r_\varepsilon z\right)}{I_0\left(\frac{1}{2}z\right)} + \frac{K_1\left(r_\varepsilon z\right)}{K_0\left(r_\varepsilon z\right)}\right). \tag{121}$$

For $r_\varepsilon^{-1/3} < z < r_\varepsilon^{-1}$, we have

$$\sqrt{\lambda}\frac{I_1\left(r_\varepsilon z\right)}{I_0\left(\frac{1}{2}z\right)} \le \sqrt{\lambda}\frac{I_1(1)}{I_0\left(\frac{1}{2}\sqrt{\lambda}\varepsilon\right)} \to 0 \text{ as } \varepsilon \to 0.$$

On the other hand, it is easy to check that for $0 < x < 1$,

$$\frac{K_1\left(x\right)}{K_0\left(x\right)} \le 2\frac{1}{x\ln\left(\frac{e}{x}\right)}.$$

We thus obtain for $z = r_\varepsilon^{-\alpha}$, $\frac{1}{3} \le \alpha \le 1$, that

$$|\psi'(\varepsilon r_\varepsilon)| \le CM\left(1 + \frac{1}{r_\varepsilon\varepsilon}\frac{1}{\ln\left(\frac{e}{r_\varepsilon^{1-\alpha}}\right)}\right) \le CM\frac{\varepsilon}{r_\varepsilon(1-\alpha+\varepsilon^2)} = C\theta(\lambda,\varepsilon)M. \tag{122}$$

For the remaining case $z \ge r_\varepsilon^{-1}$, note that

$$I_1(r_\varepsilon x) < I_0(r_\varepsilon x) \le I_0\left(\frac{1}{2}x\right) \text{ for } x > 0,$$
$$K_1(x) \le 2K_0(x) \text{ for } x \ge 1.$$

Inserting these inequalities in (121), we obtain

$$|\psi'(\varepsilon r_\varepsilon)| \le CM\sqrt{\lambda} \text{ for } z > r_\varepsilon^{-1}. \tag{123}$$

From (117), (120), (122) and (123), we conclude that

$$\sup_{\partial D_{m,n,\varepsilon}} \frac{\partial\tilde{u}_\varepsilon}{\partial n} \le |\psi'(\varepsilon\,r_\varepsilon)| \le C\theta(\lambda,\varepsilon)\,M.$$

The lower bound for $\frac{\partial\tilde{u}_\varepsilon}{\partial n}$ is obtained by similar arguments. Since $\tilde{u}_\varepsilon$ is constant on $\partial D_{m,n,\varepsilon}$, this concludes the proof.

A.4 Proof of Lemma 10

We continue to use $M = \sup_{m,n} |u_{m,n}|$. Fix a multi-index $J \in \{(0,1),(1,0)\}$ and split $\nabla^J \tilde{u}_\varepsilon(x) = I_1^J(x) + I_2^J(x) + I_3^J(x)$ as in (95).

We first show that

$$[\nabla \tilde{u}_\varepsilon]_{C^\nu(\omega_\tau^\varepsilon)} \le C\,M \text{ for } \tau > \kappa\,\varepsilon^{(1-\nu)/2}. \tag{124}$$

Recall that we have shown using (96) and (97) with $\tilde{\beta} = 0$ that

$$|\nabla I_1^J(x)| \le CM \text{ and } |\nabla I_2^J(x)| \le \frac{CM\,r_\varepsilon}{\tau^3 \varepsilon^3}.$$

Fix x and z in ω_τ^ε and $\nu \in (0,1)$. Then the above estimates implies that

$$\frac{|I_1^J(x) - I_1^J(z)| + |I_2^J(x) - I_2^J(z)|}{|x-z|^\nu} \le C\,M\Big(\frac{r_\varepsilon}{\tau^3\,\varepsilon^3} + 1\Big).$$

To establish (124), it remains to bound

$$
\begin{aligned}
A(x,z) &:= \frac{|I_3^J(x) - I_3^J(z)|}{|x-z|^\nu} \\
&= \sum_{(m,n)\in I_\varepsilon} \int_{\partial D_{m,n,\varepsilon}} \frac{\partial \tilde{u}_\varepsilon}{\partial n}(y)\, \frac{[\nabla_x^J \Phi(x-y) - \nabla_x^J \Phi(z-y)]}{|x-z|^\nu}\, d\sigma(y).
\end{aligned}
$$

For $\lambda \ge \frac{1}{r_\varepsilon}$, thanks to (100), we have

$$A(x,z) \le C|\nabla I_3^J| = C|I_3^{J+1}| \le C\,M\,\frac{r_\varepsilon^{1/2}}{\sqrt{\lambda}\,\varepsilon^5}\Big(\frac{\varepsilon}{\tau^4} + 1\Big).$$

which is sufficient provided $\tau > \kappa\varepsilon^2$ for example. We henceforth assume that $\lambda \le \frac{1}{r_\varepsilon}$. We first bound the derivatives of I_3^J. For $\xi \in \omega_\tau^\varepsilon$, we use Lemma 9 and count contributions from "rings" of inclusions, distinguishing near and far contributions as before, to get

$$
\begin{aligned}
|\nabla I_3^J(\xi)| &\le C \sum_{j=0}^{C\varepsilon^{-1}} \theta(\lambda,\varepsilon)\,M\,(j+1)\,\lambda\,\Big|K_0^{(2)}(\sqrt{\lambda}(j+\tau)\varepsilon)\Big|\,\varepsilon\,r_\varepsilon \\
&\le C\,M\,\theta(\lambda,\varepsilon)\,\varepsilon\,r_\varepsilon\,\lambda\,\Big|K_0^{(2)}(\sqrt{\lambda}\,\tau\,\varepsilon)\Big| \\
&\quad + C\,M\,\theta(\lambda,\varepsilon)\,\varepsilon\,r_\varepsilon \sum_{j=1}^{C\varepsilon^{-1}} M\,j\,\sqrt{\lambda}^2\,\Big|K_0^{(2)}(\sqrt{\lambda}\,j\,\varepsilon)\Big| \\
&\le C\,M\,\theta(\lambda,\varepsilon)\,\varepsilon\,r_\varepsilon \left(\frac{1}{\tau^2\,\varepsilon^2} + \sum_{j=1}^{C\varepsilon^{-1}} \frac{1}{j\,\varepsilon^2}\right) \\
&\le C\,M\,\theta(\lambda,\varepsilon)\,\varepsilon\,r_\varepsilon \left(\frac{1}{\tau^2\,\varepsilon^2} + \frac{|\log\varepsilon|}{\varepsilon^2}\right).
\end{aligned}
$$

Noting that $\theta(\lambda,\varepsilon) \le \frac{\varepsilon}{r_\varepsilon}$ by (91) and our restriction on λ, we thus have

$$|\nabla I_3^J(\xi)| \le C\,M\left(\frac{1}{\tau^2} + |\log\varepsilon|\right). \tag{125}$$

From (125) we deduce in particular that

$$A(x,z) \leq C\,M\,\varepsilon^{1-\nu}\left(\frac{1}{\tau^2} + |\log\varepsilon|\right) \text{ for } |x-z| < 10\varepsilon. \tag{126}$$

We turn to bounding A when $|x-z| > 10\varepsilon$. Pick $\tilde{x}$ and $\tilde{z}$ in ω_τ^ε such that $|x-\tilde{x}| < 2\varepsilon$, $|z-\tilde{z}| < 2\varepsilon$, $\text{dist}\,(\tilde{x}, D_\varepsilon) > \varepsilon/10$, $\text{dist}\,(\tilde{z}, D_\varepsilon) > \varepsilon/10$ and $|\tilde{x}-\tilde{z}| < |x-z|$. Then by (126),

$$A(x,z) \leq A(x,\tilde{x}) + A(\tilde{x},\tilde{z}) + A(\tilde{z},z) \leq A(\tilde{x},\tilde{z}) + C\,M\,\varepsilon^{1-\nu}\left(\frac{1}{\tau^2} + |\log\varepsilon|\right).$$

Thus, provided $\tau > \kappa\varepsilon^{\frac{1-\nu}{2}}$, can focus on case when $\text{dist}\,(x, D_\varepsilon) > \varepsilon/10$ and $\text{dist}\,(z, D_\varepsilon) > \varepsilon/10$.

Split $A(x,z) = A_1(x,z) + A_2(x,z)$ where

$$A_1(x,z) = \sum_{(m,n)\in I_\varepsilon(x,z)} \int_{\partial D_{m,n,\varepsilon}} \frac{\partial \tilde{u}_\varepsilon}{\partial n}(y) \frac{[\nabla_x^J \Phi(x-y) - \nabla_x^J \Phi(z-y)]}{|x-z|^\nu}\, d\sigma(y),$$

$$A_2(x,z) = \sum_{(m,n)\in I_\varepsilon(z,x)} \int_{\partial D_{m,n,\varepsilon}} \frac{\partial \tilde{u}_\varepsilon}{\partial n}(y) \frac{[\nabla_x^J \Phi(x-y) - \nabla_x^J \Phi(z-y)]}{|x-z|^\nu}\, d\sigma(y).$$

and

$$I_\varepsilon(x,z) = \{(m,n) \in I_\varepsilon : \text{dist}\,(x, D_{m,n,\varepsilon}) \leq \text{dist}\,(z, D_{m,n,\varepsilon})\}.$$

In the sequel we bound $A_1(x,z)$ independently of x and z. Switching the role of x and z, we can therefore use the same bound for $A_2(x,z)$.

Using Lemma 9 and the expression for $\theta(\lambda,\varepsilon)$ for $\lambda \leq \frac{1}{r_\varepsilon}$, we have

$$|A_1(x,z)| \leq \sum_{(m,n)\in I_\varepsilon(x,z)} C\,M\,\varepsilon^2 \sup_{y\in\partial D_{m,n,\varepsilon}} \left|\frac{[\nabla_x^J \Phi(x-y) - \nabla_x^J \Phi(z-y)]}{|x-z|^\nu}\right|. \tag{127}$$

It is convenient to introduce

$$K_0^{(1+\nu)}(t) := \sup_{s,s'>t} \frac{|K_0^{(1)}(s) - K_0^{(1)}(s')|}{|s-s'|^\nu}.$$

We will use the following inequality, which is easily proved using the monotonicity properties of $K_0^{(1)}$ and $K_0^{(2)}$, the derivative of $K_0^{(1)}$,

$$K_0^{(1+\nu)}(t) \leq \frac{C}{t^{1+\nu}} \text{ for } t > 0. \tag{128}$$

To bound $A_1(x,z)$, we proceed as before by counting contribution from inclusions located in the rings centred at x with radii $(j+1/10)\varepsilon$ and $(j+1+1/10)\varepsilon$. For example, consider an inclusion

$D_{m,n,\varepsilon}$ which is closer to x than to z and lying in the above j-th ring. For $y \in \partial D_{m,n,\varepsilon}$, we estimate

$$
\frac{|\nabla_x^J \Phi(x-y) - \nabla_x^J \Phi(z-y)|}{|x-z|^\nu}
$$

$$
= \frac{1}{|x-z|^\nu} \left| \sqrt{\lambda}\, K_0^{(1)}(\sqrt{\lambda}\,|x-y|) \frac{(x-y) \cdot J}{|x-y|} - \sqrt{\lambda}\, K_0^{(1)}(\sqrt{\lambda}\,|z-y|) \frac{(z-y) \cdot J}{|z-y|} \right|
$$

$$
\leq \frac{1}{|x-z|^\nu}\, \sqrt{\lambda}\, |K_0^{(1)}(\sqrt{\lambda}\,|x-y|)| \left| \frac{(x-y) \cdot J}{|x-y|} - \frac{(z-y) \cdot J}{|z-y|} \right|
$$

$$
+ \frac{1}{|x-z|^\nu} \left| \sqrt{\lambda}\, K_0^{(1)}(\sqrt{\lambda}\,|x-y|) - \sqrt{\lambda}\, K_0^{(1)}(\sqrt{\lambda}\,|z-y|) \right| \left| \frac{(z-y) \cdot J}{|z-y|} \right|
$$

$$
\leq \sqrt{\lambda}\, |K_0^{(1)}(\sqrt{\lambda}\,(j+1/10)\,\varepsilon)| \frac{C}{((j+1/10)\varepsilon)^\nu} + \sqrt{\lambda}^{1+\nu}\, K_0^{(1+\nu)}(\sqrt{\lambda}\,(j+1/10)\,\varepsilon)
$$

$$
\leq \frac{C}{((j+1/10)\,\varepsilon)^{1+\nu}}.
$$

Inserting this estimate in (127) and summing over all inclusions we end up with

$$
A_1(x,z) \leq \sum_{j=0}^{C\varepsilon^{-1}} j\, \frac{C\, M\, \varepsilon^{1-\nu}}{(j+1/10)^{1+\nu}} \leq C\, M\, \varepsilon^{1-\nu} \sum_{j=1}^{10C\varepsilon^{-1}} \frac{1}{j^\nu} \leq C\, M.
$$

Likewise

$$
A_2(x,z) \leq C\, M.
$$

We have thus established (124).

To establish the result, we note that, by Lemma 7, we have, for $0 < \tilde{\beta} < \frac{1}{2}$ and $\tau > \kappa\, \varepsilon^{\frac{1-2\tilde{\beta}}{2(1+2\tilde{\beta})}}$,

$$
\|\nabla \tilde{u}_\varepsilon\|_{L^\infty(\omega_\tau^\varepsilon)} \leq \frac{C}{\lambda^{\tilde{\beta}}}\, M.
$$

This implies that

$$
\frac{|\nabla \tilde{u}_\varepsilon(x) - \nabla \tilde{u}_\varepsilon(z)|}{|x-z|^\nu} \leq \frac{C}{\lambda^{\tilde{\beta}}} \frac{M}{|x-z|^\nu}.
$$

On the other hand, for some small $\delta > 0$ to be determined, (124) implies that for $\tau > \kappa\, \varepsilon^{\frac{\delta}{2}}$ that

$$
\frac{|\nabla \tilde{u}_\varepsilon(x) - \nabla \tilde{u}_\varepsilon(z)|}{|x-z|^\nu} \leq [\nabla \tilde{u}_\varepsilon]_{C^{1-\delta}(\omega_\tau^\varepsilon)}\, |x-z|^{1-\delta-\nu} \leq C\, M\, |x-z|^{1-\delta-\nu}.
$$

It follows that, provided $0 < 1-\delta-\nu$,

$$
\frac{|\nabla \tilde{u}_\varepsilon(x) - \nabla \tilde{u}_\varepsilon(z)|}{|x-z|^\nu} \leq \left\{ \frac{C}{\lambda^{\tilde{\beta}}} \frac{M}{|x-z|^\nu} \right\}^{\frac{1-\delta-\nu}{1-\delta}} \left\{ C\, M\, |x-z|^{1-\delta-\nu} \right\}^{\frac{\nu}{1-\delta}}
$$

$$
= \frac{C}{\lambda^{\frac{\tilde{\beta}(1-\delta-\nu)}{1-\delta}}}\, M \text{ provided } \tau > \kappa\, \varepsilon^{\min\left(\frac{\delta}{2}, \frac{1-2\tilde{\beta}}{1+2\tilde{\beta}} \right)}.
$$

Picking $\tilde{\beta} = \frac{\beta - \frac{\nu}{4}}{1 - \frac{\nu}{2}}$ and $\delta = \frac{1-2\tilde{\beta}}{1+2\tilde{\beta}} = \frac{1-2\beta}{1+2\beta-\nu}$, we get the assertion, provided $\nu < 2\beta$.

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